Formosa Journal of Applied Sciences (FJAS)
Vol. 2, No. 5 2023: 823-836

Numerical solution of a Volterra integral equation

Ahmed Abdulkareem Hadi
Mathematics Science, Ministry of Education, Directorate of Education Baghdad Al-Rusafa
Corresponding Author: Ahmed Abdulkareem Hadi
ahmed08hadi@gmail.com

A R T I C L E I N F O

Keywords: Numerical Solution, Volterra, Integral Equation, Second Kind, Hermite Polynomials

Received : 21, March
Revised : 23, April
Accepted: 25, Mei

A B S T R A C T

This work deals with a generalized nonlinear Volterra integral equation of the second kind, in whose kernel the unknown function occurs with two different arguments. The equation is solved by a collocation approach with piecewise Hermite polynomials. When using polynomials of degree 2 m- 1, m e N, and suitable quadrature formulas, the method has the order 2 m. The collocation points must be chosen in accordance with a stability condition.

INTRODUCTION

Volterra integral equations are a special type of integral equations that involve integration over the past history of the unknown function. They are divided into two groups referred to as the first and the second kind. A linear Volterra equation of the first kind is

$$f(x) = g(x) + \int_{0}^{x} K(x, y, f(y), f(a(x, y)))\, dy. \qquad (1)$$

where f and K are given functions and x is an unknown function to be solved for. Volterra integral equations of the second type, In the classical theory of E. I. Fredholm, X was the Banach space of continuous functions on a closed interval. Another natural situation, which however offers some additional difficulties, is the case X = L_2(a, b) functions of square-integrable functions on a (closed) interval. Finally, there is a case where X is a space of number sequences and T is an infinite matrix. In this case, one can see that this is a linear system of equations in infinitely many unknowns and with infinitely many equations.

Tahmasbi et al. proposed a numerical method for solving systems of linear Volterra integral equations of the second kind based on the power series method. This method does not require derivatives and can recover the analytical solution when it is a polynomial. The numerical results showed that the method is simple and effective. they used MATLAB for the numerical computations in their study(Tahmasbi & Fard, 2008).

Steinberg applied Gregory's formula to numerically integrate Volterra linear integral equations of the second kind. He used recursive inequalities to derive the order and the asymptotic behavior of the truncation error (Steinberg, 1972).

Salim el al. used the linear spline function of the unknown function at an arbitrary point to numerically solve linear mixed Volterra-Fredholm integral equations of the second kind. they transformed the integral equation into a system of linear equations with respect to the unknown function by integrating. They can obtain an approximate solution by solving this system with a Python code program version 3.9. they also prove theoretical results on the uniqueness and convergence of the method (Salim, Saeed, & Jwamer, 2022).

Fahim et al. numerically solved a Volterra integro-differential equation of parabolic type with memory term and initial boundary value conditions. They used finite difference method and product trapezoidal integration rule for time discretization and sinc-collocation method for space discretization. they focused on the case of a weakly singular kernel. they showed that the method converges exponentially to the solution and provide the convergence analysis. they also gave numerical examples and illustrations to support the method (Fahim, Fariborzi Araghi, Rashidinia, & Jalalvand, 2017). A Volterra integro-differential equation of parabolic type with memory term and initial boundary value conditions solved numerically. Finite difference method and product trapezoidal integration rule have used for time discretization and sinc-collocation method for space discretization. We focus on the case of a weakly singular kernel. The method converges exponentially to the solution have shown and provided the convergence analysis. The numerical examples and illustrations to support the method also have given.

Lema and co-workers studied second kind Volterra integral equations with weakly singular kernels. They defined some suitable function spaces and showed that Euler's method has an asymptotic error expansion. This result enables them to apply some extrapolation methods, which they successfully demonstrate with some numerical examples (Lima & Diogo, 1997).

In this work we describe a method for the numerical solution of the integral equation(Steinberg, 1972):

$$f(x)=g(x)+\int_0^x K(x,y,f(y),f(a(x,y)))\,dy. \qquad (1)$$

Equations of a similar type occur when evaluating experiments in the Atomic Physics on (Doornenbal, 2012; Zhu, Ma, Chen, & Han, 2016). We generally make the following assumption:

(i) $A>0,\ I=[0,A],\ G=\{(x,y)\in\mathbb{R}^2/0\le y\le x\le A\},$
$g\in C(I),\ \ K\in C(G\times\mathbb{R}\times\mathbb{R}),\ \ a\in C(G).$

(ii) There is constants $L_1, L_2 > 0$ so for all

$$(x,y,z_1,z_2),\ (x,y,\bar z_1,\bar z_2)\in G\times\mathbb{R}\times\mathbb{R}\ \text{gilt} \qquad (2)$$
$$|K(x,y,z_1,z_2)-K(x,y,\bar z_1,\bar z_2)|\le L_1|z_1-\bar z_1|+L_2|z_2-\bar z_2|.$$

(iii) There exist $k\in[0,1)$ and $b > 0$ with:

$$0\le a(x,y)\le\max(kx,\ x-b),\ (x,y)\in G.$$

To solve (1) we will use a collocation method (see references Delves and Walsh (Golberg, 2013) and Atkinson (Atkinson, 1972)). Hermite functions (see Ciarlet, Schultz and Varga (Ciarlet, Schultz, & Varga, 1967; Jajarmi & Baleanu, 2020)), i.e. functions that (m - 1) times are continuously differentiable and piecewise with polynomials of degree 2 m – 1. The approximate solution can be calculated step by step as in an initial value problem(Niu, Lin, & Zhang, 2012).

Piecewise polynomials have been used several times to solve the integral equation:

$$f(x)=g(x)+\int_0^x K(x,y,f(y))\,dy \qquad (3)$$

The procedure of Tom [5] is a special case of the method explained here. Netravali (Netravali, 1973) solves (3) with twice continuously differentiable cubic splines. De Hoog and Weiss (Diogo, McKee, & Tang, 1994) make an overall collocation approach for (3) with piecewise polynomials that are only continuous. In this essay, all statements and procedures for a scalar integral equation are formulated for reasons of clarity. However, they apply without distinction also for systems of integral equations of type (1) (i.e.[(Maleknejad & Aghazadeh, 2005)]).

Existence statements

We quote some of the existence statements proved in (Maleknejad & Aghazadeh, 2005) for the integral equation (1). Before that we note that (1) has no solution in general in C *(I)* if condition (2 iii) is not satisfied. For example, the equation:

$f(x)=1+\int_0^x c f(x)\,dy$: $f(x)=(1-cx)^{-1}.$

LITERATURE REVIEW

Theorem 1 :

i. Under the assumptions (2) the integral equation (1) has a unique one solution f □ C (I).
ii. f can be estimated by a suitable c > 0 independent of A:
iii. $|f(x)|\leq e^{c(x+1)^2}\cdot\left(\max_{0\leq y\leq x}|g(y)|+x\cdot\max_{0\leq t\leq y\leq x}|K(y,t,0,0)|\right).$
iv. If for all $(x,y)\in G\ a_y(x,y)\geq\beta>0$, one can replace $e^{c(x+1)^2}$ by e^{cx} in the last estimate.
v. (iv) When $g\in C_n(I),\ K\in C_n(G\times\mathbb{R}\times\mathbb{R})$ with n > 2, then $f\in C_n(I)$.

The same is true for n = 1 under the additional assumptions: K_x *and Kz* satisfy \ Lipschitz conditions in a neighborhood of the point (0, 0, g (0), g (0)).

Regarding the last argument.

(v) Under the conditions of (iv) applies

$$f^{(n)}(x)=g_n(x)+\int_0^x K_n(x,y)\,f^{(n)}\big(a(x,y)\big)\,dy.$$

In g_n and K_n, g and K occur with their derivatives as well as *f, ...,f(n-1).*
In this and the following sections we use the following notations:

$$m\in\mathbb{N},\ \ N\in\mathbb{N},\ \ h=A/N,\ \ x_n=n\,h,\ \ n=0\,(1)\,N.$$

We cite some facts about piecewise Hermite interpolation.

1. **Hermite – Functions**

In this and the following sections, we use the following designations(Weisner, 1959):

$$m\in\mathbb{N},\ \ N\in\mathbb{N},\ \ h=A/N,\ \ x_n=n\,h,\ \ n=0\,(1)\,N.$$

We cite some facts about the staggered Hermite interpolation(Imai & Aoki, 2004).

Theorem 2:

(i) Let f ∈ C_{m_1} (I), then there exists exactly one function *P_m (. ;f, h)* C_{m_1} (I), which is in the subintervals [x_{n-1}, x_n], n = l (1)N, with a polynomial yore degreeless than or equal to 2 m - 1 and which satisfies the interpolation condition filled.

$$P_m^{(j)}(x_n; f, h) = f^{(j)}(x_n),\ j=0\,(1)\,m-1,\ n=0\,(1)\,N,$$

(ii) Let f ∈ Cm_ 1 (I), then then for j=0 (1) 2 m - 1 applies

$$\| f^{(j)} - P_m^{(j)}(\cdot; f, h) \|_\infty \le H_{mj}\, h^{2m-j}\, \| f^{(2m)} \|_\infty,$$

$$H_{mj} = \begin{cases} \dfrac{m^m (m-j)^{m-j}}{(2m-j)^{2m-j}} \dfrac{1}{(2m-j)!}, & j=0\,(1)\,m-1, \\ \dfrac{1}{(2m-j)!}, & j=m\,(1)\,2m-1. \end{cases}$$

He error estimate in the form given can be found in Kansy (Kansy, 1973; Levesley & Luo, 2003).Very general error statements even for lower smoothness properties off can be found in the work of Swartz and Varga (Swartz & Varga, 1972).

The Hermite interpolating P_m (. ; f, h) of f can bet as a linear combination of basic functions R_{m}j (x; x_n, h), j = 0 (1) m - 1, n = 0 (1) N:

$$P_m(x; f, h) = \sum_{n=0}^{N} \sum_{j=0}^{m-1} f^{(j)}(x_n)\, R_{mj}(x; x_n, h),\quad x \in I.$$

Lemma 1 : For j = 0 (1) m - 1 and n = 0 (1) N or n = 1 (1) N holds:

(i) R_{mj}(X;Xn, h)=0 , if | | x - x_n | | > h .

(ii) Let θ∈ [0, 1], x = x_n + θ h , then:

$$R_{mj}(x; x_{n-1}, h) = h^j R_{mj}(\theta; 0, 1),$$

$$R_{mj}(x; x_n, h) = h^j R_{mj}(\theta, 1, 1).$$

(iii) There exists a constant p_m > 0 independent of h with:

$$\max_{x \in I} | R_{mj}^{(l)}(x; x_n, h) | \le h^{j-l} \rho_m,\quad l=0\,(1)\,m-1.$$

With the help of the basic functions, one defines the Hermite function P_m(x; u, h) to a vector:

$$u = (u_{nj})_{n=0\,(1)\,N,\ j=0\,(1)\,m-1} \in \mathbb{R}^{m(N+1)}$$

Where

$$P_m(x; u, h) = \sum_{n=0}^{N} \sum_{j=0}^{m-1} u_{nj}\, R_{mj}(x; x_n, h),\quad x \in I.$$

It applies

$$P_m^{(j)}(x_n; u, h) = u_{nj}, \quad j = 0\,(1)\,m-1,\ n = 0\,(1)\,N.$$

METHODOLOGY

General nonlinear Volterra integral methods of the second kind, where an unknown function kernel appears with two different arguments.

RESEARCH RESULT AND DISCUSSION

As an approximation for the solution f of the integral equation (1)(Dzyadyk, 1995), we determine a Hermite function $P_m(x;u,h)$ at the vertices x_n, n=0(1)N. The components of the vector $u=(u_{nj})$ then form approximations for the function and derivative values of f at the points of intersection:

$$u_{nj} \approx f^{(j)}(x_n), \quad j = 0\,(1)\,m-1,\ n = 0\,(1)\,N.$$

We call the vector u = u (m, h) the discrete solution of (1).

To calculate $P_m(x; u,h)$, we first formally insert this Hermite function intoformally into (1). Then we replace the integral by one that fits the subintervals $[x_{n-1}, x_n]$, n = l (1)N, matching the summed quadrature formula. By inserting of m arguments x_{nj}, j = 0 (1) m - 1, per subinterval $[x_{n-1}, x_n]$we then obtain m.N equations for the coefficients u_{nj} ("collocation"). The starting coefficients U_{oo}.... ,$u_{o,m-1}$ are calculated directly from (1). For each partial interval a (generally non-linear) m x m equation system is obtained. To obtain a procedure of order 2m we must use Hermite functions of degree 2 m – 1 and integration formulae of order 2 m.

The summed quadrature formula should be constructed as follows. For h>0, φ∈ C_{2m} ([0, h]) let a quadrature formula be given by:

$$\int_0^h \varphi(y)\,dy = h \sum_{l=1}^{q} W_l\, \varphi(d_l\, h) + E_0(\varphi),$$

d1 are the parameters of the points, W_l the weights.Practically, one will only use formulae with $W_l > 0$. Let $E_o(\varphi)$ be deductible from:

$$|E_0(\varphi)| \leq M_Q \cdot \max_{0 \leq y \leq h} |\varphi^{(2m)}(y)| \cdot h^{2m+1}.$$

To approximate the integral

$$\int_0^x \varphi(y)\,dy, \quad x \in (x_{n-1}, x_n],$$

this formula becomes summation:

$$\int_0^x \varphi(y)\,dy = \sum_{i=1}^{n-1} \int_{x_{i-1}}^{x_i} \varphi(y)\,dy + \int_{x_{n-1}}^{x} \varphi(y)\,dy$$

$$= \sum_{i=1}^{n-1} h \sum_{l=1}^{q} W_l\, \varphi(x_i + d_l\, h) + (x - x_n) \sum_{l=1}^{q} W_l\, \varphi(x_n + d_l\, h) + E(\varphi; x)$$

$$= \mathbf{J}_0^x \varphi(y)\,dy + E(\varphi; x).$$

The following then applies as:

$$|E(\varphi; x)| \leq M_Q \cdot x \cdot \max_{0 \leq y \leq x} |\varphi^{(2m)}(y)| \cdot h^{2m}.$$

Suitable quadrature formulae in this sense are the Gauss-formulae with m points or the Lobatto formulae with m + 1 points. Other formulas require more function evaluations.

We can now define the procedure of order 2 m with N subintervals.

1. Set h=A/N, x_n=nh, n = 0 (1) N . Choose θo, ..., θm ∈ [0, 1] with 0 < θo <….< θ_{m-1} < 1.
   Set x_{nj}=X_{n-1}+Ojh, j = 0 (1) m - 1 , n = l (1)N.
2. Calculate the values $f^{(J)}$(0), j = 0 (1) m - 1 , by differentiation of the integral equation and set u_{oj}=$f^{(j)}$ (0).
3. Let n ∈ {1..... N} and let u_{lj} be calculated for l<n-1. Calculate U_{no} ..... u_{m-1} from the equations

$$P_m(x_{nj}; u, h) = g(x_{nj}) + \quad (5)$$
$$+ \mathbf{J}_0^{x_{nj}} K\left(x_{nj}, y, P_m(y; u, h), P_m\left(a(x_{nj}, y); u, h\right)\right) dy, \quad j = 0\,(1)\,m-1.$$

Remarks:

Practically , the methods for m = 1, 2, 3, possibly m = 4.
The choice of the parameters 0iis not quite arbitrary. For stability reasons they are , in dependence of m , subject to certain restrictions . Normally one will choose θ_{m-1}=1.
For the first derivatives of f in zero one gets:

$$f\ (0) = g(0)$$
$$f'\ (0) = g'(0) + K\left(0, 0, f(0), f(0)\right)$$
$$f''(0) = g''(0) + 2 \cdot K_x\left(0, 0, f(0), f(0)\right)$$
$$+ K_y\left(0, 0, f(0), f(0)\right) + K_{z_1}\left(0, 0, f(0), f(0)\right) f'(0)$$
$$+ K_{z_2}\left(0, 0, f(0), f(0)\right) \cdot \left(2\, a_x(0,0) + a_y(0,0)\right) \cdot f'(0).$$

The unknowns U_{no}, ... , $U_{n,m-1}$ occur on the left side, in the argument P_m(y;u,h) at the last subinterval and in the argument P_m(a($x_{n,j}$,y);u,h) at the subintervals where a(x_{nj}, y) > x_{n-1}.

The $m \times m$ system of equations (5) can be solved by Newton's method. Initial values for the iterations are $U_{n-1,0}$, ... , $U_{n,m-1}$ The functional matrixfor the Newton method can be calculated exactly or also be formed by numerical differentiation.
If the constants θj are chosen in such a way that the stability condition (12) is satisfied, the equations (5) for a sufficiently small h are uniquely sable.

The stability condition

In this section, we want to discuss the stability condition(Burton & Zhang, 2004):

$$\rho(R^{-1} Q) < 1 \qquad (12)$$

This condition restricts the choice of the intermediate points $\theta = (\theta_0, \ldots, \theta_{m-1}) \in \mathbb{R}^m$, for which the condition $0 < \theta_0 < \ldots < \theta_{m-1}$ was set. The following example shows that such a restriction is useful.

If one applies to the integral equation:

$$f(x) = 1, \quad x \in I,$$

our method with m = 1 and $\theta_0 \in (0, 1]$, one obtains for the zeros u_n of the exact solution *f (n h)= 1* the recursion:

$$u_n = \frac{\theta_0 - 1}{\theta_0} u_{n-1} + \frac{1}{\theta_0}, \quad n \in \mathbb{N}, \quad u_0 = 1,$$

so $u_n = 1$ for all $n \in \mathbf{N_0}$. The same recursion with initial value $\tilde{u}_0 = 1 + \varepsilon$, has the solution

$$\tilde{u}_n = 1 + \varepsilon \left(\frac{\theta_0 - 1}{\theta_0} \right)^n, \quad n \in \mathbb{N}_0.$$

For $\theta_0 \in (0, 1]$, the perturbation with $h \to 0$, where $n \to \infty$ is stable for all limits, the discretisation is not stable.

Before we give the practically important cases of the number θ, we quote a theorem about the existence of stable discretisation.

Theorem 3

For $m \in \mathbf{N}$ and d >1, Φ_d the set of all $\theta = (\theta_0, \ldots, \theta_{m-1}) \in \mathbf{R^m}$ with $0 < \theta_0 < \ldots\ldots < \theta_{m-1} < 1$ and :

$$\max_{i=1}^{m} (\theta_i - \theta_{i-1}) \Big/ \min_{i=1}^{m-1} (\theta_i - \theta_{i-1}) \leq d.$$

Then there exists a constant D (m, d) $\in [0, 1)$, so that for all $\theta \in \Phi_d$ d with $\theta o > D(m, d)$ the condition $p(R^{-1} Q) < 1$ is fulfilled.

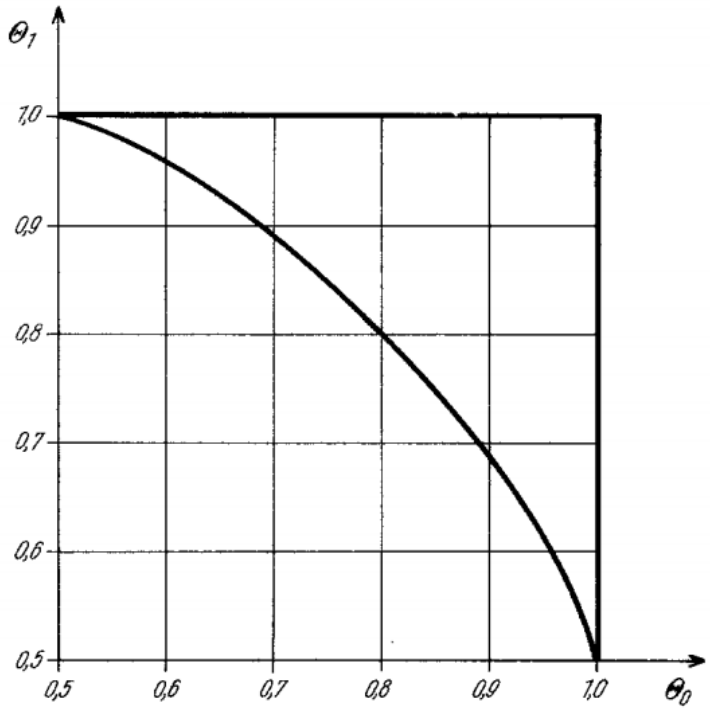
Proof: The statement can be shown with the help of a deduction for the inverse of the Vandermonde-matrix.

In the cases m = 1, 2, 3 admissible θ can be determined directly from condition (12).

m = 1; The stability condition is fulfilled if $\theta_0 > 1/2$.

m = 2; The stability condition is fulfilled if $1/2 < \theta_0 < 1$ and $\theta_1 = 1$.

m = 3: The stability range was determined by numerical calculation of p ($R^{-1}Q$) (see. [6]). The result is shon in Fig. 1: For m=3 the stability condition is fulfilled, if θ_2=1 and ($\theta 0_1$, θ_1) in the strongly bordered area of the square [0.5, 1] x [0.5, 1].



Picture.1 Vandermonde-matrix.

In the practical calculations, good experience has been made with the following values for θ.

$$m=1: \ \theta_0=1$$
$$m=2: \ \theta_0=0.6, \ \theta_1=1$$
$$m=3: \ \theta_0=0.75, \ \theta_1=0.87, \ \theta_2=1$$

Generalisation

In reference Meis (Groetsch; Meis, 1978), they dealt with the following generalisation of (1):

$$f(x)=F\left(x, f(d(x)), \int_0^x K(x, y, f(y), f(a(x, y)))\, dy\right), \quad x \in I, \qquad (13)$$

With

$$F \in C(I \times \mathbb{R} \times \mathbb{R}),\ K \in C(G \times \mathbb{R} \times R),$$
$$a \in C(G),\ \ 0 \leq a(x, y) \leq \max(kx, x-b),\ \ (x, y) \in G,$$
$$d \in C(I),\ \ 0 \leq d(x) \ \ \leq \max(kx, x-b),\ \ x \in I.$$

Meis shows, under suitable Lipschitz conditions on F and K, that (13) has an unambiguous, continuous solution. Furthermore, under certain conditions the equation of the first kind:

$$0 = g(x) + \int_0^x K(x, y, f(y), f(a(x, y)))\,dy,\ \ x \in I, \tag{14}$$

with g ∈ C (I), other conditions as above, to a system of two equations (13).

The numerical method described above has been extended by Maskos [-9]. in such a way that it can handle systems of equations of type (13) and the equation (14) to such systems. The numerical results confirm the expectation that the convergence statements can also be applied to the more general procedure.

Numerical Results

The effectiveness of the procedure is to be demonstrated by means of two examples. The calculations were carried out inhouse computer in the university. The machine accuracyis about 14 decimal. For θ, the values given at the end of section 5were used. The Gaug formulae were used for squaring.
Example 1: (system for 2 equations of type (1)):

$$f_1(x) = 1 + \int_0^x e^{-a(x, y)} \cdot f_1(a(x, y)) \cdot \sqrt{\frac{f_1(y)}{f_2(y)}}\,dy$$

$$f_2(x) = e^{-x} + \tfrac{x}{2}\,a(x, 0) + \int_0^x f_1(y) f_2(y) \log(f_2(a(x, y)))\,dy$$

With

$$a(x, y) = 0.9\,(x-y)\cos^2 2x.$$

The Solution:

$$f_1(x) = e^x,\ \ f_2(x) = e^{-x}$$

The solution was approximated in the interval I = [0,4]. In the following tables, the maximum relative errors in I for the approximation function and its derivatives - as far as they are derivatives - are given .

Tables.1 The Maximum Relative Errors in I for the Approximation Function and its Derivatives

h	m=1		m=2			
	f1	f2	f1	f2	f '1	f '2
0.8			7 . 1' - 3	4 . 4 ' - 2 2	2 . 6 ' - 2	4 . 8 ' - 1
0.4			3.3' - 4	1.8'-3	3 . 9 ' - 3	4.0' - 2
0.2	1.4' - 1	8 .2 ' - 1	2 . 0 ' - 5	1.9' - 4	3.7' - 4	6 . 7 ' - 3
0.1	2.0' - 2	2.0' - 1	1.7' - 6	1 . 5 ' - 5	4 . 2 ' - 5	5.5' - 4
0.05	4 .8' - 3	4.4' - 2	1,1' - 7	7.5' - 7	5 . 9 ' - 6	5 . 1 ' - 5

Maximum relative error in I, m = 1, m = 2

	f1	f2	f'1	f'2	f ''1	f ''2
0.8	6.4' - 5	6.4' - 5	6.4' - 5	6.4' - 5	6.4' - 5	6.4' - 5
0.4	2 . 8 ' - 6	2 . 8 ' - 6	2 . 8 ' - 6	2 . 8 ' - 6	2 . 8 ' - 6	2 . 8 ' - 6
0.2	1.9' - 8	1.9' - 8	1.9' - 8	1.9' - 8	1.9' - 8	1.9' - 8
0.1	7 . 8 ' - 11	7 . 8 ' - 11	7 . 8 ' - 11	7 . 8 ' - 11	7 . 8 ' - 11	7 . 8 ' - 11
0.05	1 . 8 ' - 12	1 . 8 ' - 12	1 . 8 ' - 12	1 . 8 ' - 12	1 . 8 ' - 12	1 . 8 ' - 12

Maximum relative error in I, m = 3

Example 2 (equation of type (13), taken from [9]):

$$f(x)=\frac{3}{8}x+\frac{1}{16}\sin(2x)\left(4-\cos(2x)\right)+\frac{5}{6}\sin x+\frac{1}{2}f\left(\frac{x}{3}\right)-\frac{2}{3}\left(f\left(\frac{x}{3}\right)\right)^3-$$
$$-\int_0^x (f(y))^4\cdot\left(\left(f\left(\frac{xy}{2}\right)\right)^2+\frac{1}{2}(1+\cos(xy))\right)dy.$$

The solution:

$$f(x)=\sin x.$$

Tabel.2 The solution was approximated in the interval I = [0, 1]

h	m=1	m=2		m=3		
	f	f	f '	f	f '	f ''
0.5	3 . 8 ' - 2	2.0' - 4	6.1' - 3	3 . 1 ' - 6	5.3' - 5	5.9' - 4
0.25	1.2' - 2	1 . 8 ' - 5	1 . 0 ' - 3	1 , 0 ' - 7	7 . 1 ' - 7	3.2' - 5
0.125	3.0''-3	1.7' - 6	1 . 5 ' - 4	1 . 8 ' - 9	1 . 0 ' - 8	2.0' - 6
0.0625	7.6' - 4	1 . 2 ' - 7	2 . 0 ' - 5	2 . 9 ' - 1 1	1 . 8 ' - 10	1.3' - 7
0.03125	1 . 9 ' - 4	7.7' - 9	2.4' - 6	6 . 7 ' - 13	4.4' - 11	1 . 5 ' - 8

Maximum relative error in I

CONCLUSIONS AND RECOMMENDATIONS

The effectiveness of the procedure is to be demonstrated by means of two examples. The calculations were carried out inhouse computer in the university. The machine accuracyis about 14 decimal. For θ, the values given at the end of section 5were used. The Gaug formulae were used for squaring.
Example 1: (system for 2 equations of type (1)):

ADVANCED RESEARCH

Still doing more advanced research to look deeper for the numerical solution of the Volterra integral equation

REFERENCES

Atkinson, K. (1972). The numerical solution of Fredholm integral equations of the second kind with singular kernels. *Numerische Mathematik, 19*(3), 248-259. doi:10.1007/BF01404695

Burton, T. A., & Zhang, B. (2004). Fixed points and stability of an integral equation: nonuniqueness. *Applied Mathematics Letters, 17*(7), 839-846.

Ciarlet, P. G., Schultz, M. H., & Varga, R. S. (1967). Numerical methods of high-order accuracy for nonlinear boundary value Problems. *Numerische Mathematik, 9*(5), 394-430. doi:10.1007/BF02162155

Diogo, T., McKee, S., & Tang, T. (1994). Collocation methods for second-kind Volterra integral equations with weakly singular kernels. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 124*(2), 199-210.

Doornenbal, P. (2012). In-beam gamma-ray spectroscopy at the RIBF. *Progress of Theoretical and Experimental Physics, 2012*(1), 03C004. doi:10.1093/ptep/pts076

Dzyadyk, V. K. (1995). *Approximation methods for solutions of differential and integral equations*: De Gruyter.

Fahim, A., Fariborzi Araghi, M. A., Rashidinia, J., & Jalalvand, M. (2017). Numerical solution of Volterra partial integro-differential equations based on sinc-collocation method. *Advances in Difference Equations, 2017*(1), 362. doi:10.1186/s13662-017-1416-7

Golberg, M. A. (2013). *Numerical solution of integral equations* (Vol. 42): Springer Science & Business Media.

Groetsch, C. W. (2007). *Integral equations of the first kind, inverse problems and regularization: a crash course*.

Imai, Y., & Aoki, T. (2004). Fourth-order accurate IDO scheme using gradient-staggered interpolation. *JSME International Journal Series B Fluids and Thermal Engineering, 47*(4), 681-689.

Jajarmi, A., & Baleanu, D. (2020). A New Iterative Method for the Numerical Solution of High-Order Non-linear Fractional Boundary Value Problems. *Frontiers in Physics, 8*.

Kansy, K. (1973). Elementare Fehlerdarstellung für Ableitungen bei der Hermite-interpolation. *Numerische Mathematik, 21*(4), 350-354. doi:10.1007/BF01436389

Levesley, J., & Luo, Z. (2003). Error estimates for Hermite interpolation on spheres. *Journal of Mathematical Analysis and Applications, 281*(1), 46-61. doi:https://doi.org/10.1016/S0022-247X(02)00451-1

Lima, P., & Diogo, T. (1997). An extrapolation method for a Volterra integral equation with weakly singular kernel. *Applied Numerical Mathematics, 24*(2), 131-148. doi:https://doi.org/10.1016/S0168-9274(97)00016-0

Maleknejad, K., & Aghazadeh, N. (2005). Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method. *Applied Mathematics and Computation, 161*(3), 915-922.

Meis, T. (1978, 1978//). *Eine spezielle Integralgleichung erster Art.* Paper presented at the Numerical Treatment of Differential Equations, Berlin, Heidelberg.

Netravali, A. N. (1973). Spline approximation to the solution of the Volterra integral equation of the second kind. *mathematics of computation, 27*(121), 99-106.

Niu, J., Lin, Y. Z., & Zhang, C. P. (2012). Approximate solution of nonlinear multi-point boundary value problem on the half-line. *Mathematical Modelling and Analysis, 17*(2), 190-202.

Salim, S., Saeed, R., & Jwamer, K. (2022). SOLVING VOLTERRA-FREDHOLM INTEGRAL EQUATIONS BY LINEAR SPLINE FUNCTION. *9*, 99-107.

Steinberg, J. (1972). Numerical solution of Volterra integral equation. *Numerische Mathematik, 19*(3), 212-217. doi:10.1007/BF01404691

Swartz, B. K., & Varga, R. S. (1972). Error bounds for spline and L-spline interpolation. *Journal of approximation theory, 6*(1), 6-49.

Tahmasbi, A., & Fard, O. S. (2008). Numerical solution of linear Volterra integral equations system of the second kind. *Applied Mathematics and Computation, 201*(1), 547-552. doi:https://doi.org/10.1016/j.amc.2007.12.041

Weisner, L. (1959). Generating functions for Hermite functions. *Canadian Journal of Mathematics, 11*, 141-147.

Zhu, L., Ma, Y.-G., Chen, Q., & Han, D.-D. (2016). Multilayer Network Analysis of Nuclear Reactions. *Scientific Reports, 6*(1), 31882. doi:10.1038/srep31882