Application of Minimum Bayes Factor to a Balanced Two-Way Anova with Random Effects

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ABSTRACT

It is a common practice in statistical analysis to draw conclusions based on significance. P-values often reflect the probability of incorrectly concluding that a null hypothesized model is true; they do not provide information about other types of error that are also important for interpreting statistical results. Standard model selection criteria and test procedures are often inappropriate for comparing models with different numbers of random effects, due to constraints on the parameter space of the variance components. In this paper, we focused on a minimum Bayes factor proposed by Held and Ott (2018) and applied it to a balanced two way analysis of variance (ANOVA) with random effects under three cases namely: **Case 1**: both factors are fixed; **Case 2**: both factors are random; **Case 3**: factor A is fixed and factor B is random. We realized that in all the three cases, considered the Bayes factor indicates weak evidence against the null hypothesis of zero variability in the effects of the levels of the factors as well as the interactions. This result is due to the conservative nature of the minimum Bayes factor.

INTRODUCTION

A typical fixed-effect analysis of an experiment involves comparing treatment means to one another in an attempt to detect differences. In a random-effects model, *the treatment levels themselves are a random sample* drawn from a larger population of treatment levels. In the latter, the objective of the researcher is to extend the conclusions based on a sample of treatment levels to *all treatment levels in the population*. In fact, the null hypothesis of random-effects ANOVA is quite different from its fixed-effects counterpart.

Statistical analysis is often used to reason about scientific questions based on a data sample, with the goal of determining "which parameter values are supported by the data and which are not" (Hoenig and Heisey, 2001). P-values been the most commonly used tool to measure evidence against a hypothesis or hypothesized model are often incorrectly viewed as an error probability for rejection of the hypothesis or, even worse, as the posterior probability that the hypothesis is true. This difficulty in interpretation of p-values has been highlighted in many articles, (See Sellke,Bayarriand Berger, 2001)
The *p*-value quantifies the discrepancy between the data and a null hypothesis of interest, usually the assumption of no difference or no effect.

A Bayesian approach allows the calibration of *p*-values by transforming them to direct measures of the evidence against the null hypothesis; it also avail us an insight into how the data supports the alternative hypothesis. A *p*-value is computed under the assumption that the null hypothesis is true, so it is conditional on null hypothesis. It does not allow for conclusions about the probability of null hypothesis given the data, which is usually of primary interest.

THEORETICAL REVIEW

The Bayes factor directly quantifies whether the data have increased or decreased the odds of the null hypothesis. A better approach than categorizing a *p*-value is thus to transform a *p*-value to a Bayes factor or a lower bound on a Bayes factor, a so-called minimum Bayes factor (Goodman 1999). But many such ways have been proposed to calibrate *p*-values, and there is currently no consensus on how *p*-values should be transformed to Bayes factors.

The Bayes factor (or its logarithm) is therefore often referred to as the strength of evidence or weight of evidence (Bernardo and Smith 2000). Several Bayes factors have been proposed over time for Student t test, Analysis of Variance (ANOVA) among others. Some of the proposed Bayes factors are Bayesian Information Criterion (BIC)-Based (Wagenmakers, 2007; Masson, 2011; Faulkenberry, 2018), while some other Bayes factors are P-value based (Sellke,Bayarriand Berger, 2001; Held &Ott, 2018). In this article, we focused on the Bayes factor proposed by Held &Ott (2018) and applied it to a balanced two way ANOVA with random effects model. Evidence against a point null hypothesis was provided by small Bayes factors $BF_{10} \leq 1$, such that Bayes factors lie in the same range as *p*-values, which facilitates comparisons. To categorize such Bayes factors, Held &Ott (2016) provided a six-grade scale See

(Table 2.1), which was proposed as a compromise of the grades proposed by Jeffreys (1961) and Goodman (1999). The Held &Ott (2018) minimum Bayes factor is smaller than all the other minimum Bayes factors, even smaller than the Goodman bound.

METHODOLOGY
Two Way Anova With Random Effects

The balanced ANOVA model with two random effects is given by:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \qquad i = 1,2,...,m\,;\ \ j = 1,2,...,n \ \ \text{and}$$

$$k = 1,2,...,p \ldots\ldots\ldots(2.1)$$

where

y_{ijk} is the kth observation in the ith row and the jth column

μ is the grand mean

α_i is the ith effect of factor A

β_j is the jth effect of factor B

$(\alpha\beta)_{ij}$ is the interaction effect of ith factor A and j^{th} factor B

ε_{ijk} is error term associated with y_{ij} (residual) effects).

The following assumptions are made:

$\varepsilon_{ijk} \sim$ independently identically distributed (iid) Normal $(0, \sigma^2)$

$\alpha_i \sim$ independently identically distributed (iid) Normal $(0, \sigma_\alpha^2)$

$\beta_j \sim$ independently identically distributed (iid) Normal $(0, \sigma_\beta^2)$

$(\alpha\beta)_{ij} \sim$ independently identically distributed (iid) Normal $(0, \sigma_{\alpha\beta}^2)$

$\varepsilon_{ijk},\quad \alpha_i,\quad \beta_j$ and $(\alpha\beta)_{ij}$ are all independent of each other

The covariance structure of the response is given by:

$$Cov\left(Y_{ijk}, Y_{i^1 jk^1}\right) = \sigma_\beta^2 \,\forall i \neq i^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.2)$$

$$Cov\left(Y_{ijk}, Y_{ij^1 k^1}\right) = \sigma_\alpha^2 \,\forall j \neq j^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.3)$$

$$Cov\left(Y_{ijk}, Y_{ijk^1}\right) = \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_{\alpha\beta}^2 \, \forall k \neq k^1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2.4)$$

$$Cov\left(Y_{ijk}, Y_{i^1 j^1 k^1}\right) = 0 ; \ i \neq i^1, \ j \neq j^1, \ k \neq k^1 \dots\dots\dots\dots\dots\dots\dots\dots(2.5)$$

In such a balanced variance components model (2.1), Bayesians are often interested in evaluating whether the random effects should be included; this is equivalent to testing the following null hypotheses:

$$H_{01} : \sigma_\alpha^2 = 0 \hspace{4cm} \text{against}$$

$$H_{11} : \sigma_\alpha^2 \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2.6 \hspace{1cm} H_{02} : \sigma_\beta^2 = 0$$

$$\text{against} \hspace{1cm} H_{12} : \sigma_\alpha^2 \neq 0 \hspace{2cm} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2.7)$$

$$H_{03} : \sigma_{\alpha\beta}^2 = 0 \text{ against } H_{13} : \sigma_{13}^2 \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2.8)$$

The frequentist approach to testing the hypotheses stated above is summarized in **Tables (**2.1), (2.2) and (2.3) below:

Table 2. Summary Table For Two Way ANOVA With Both Factors Fixed

Source of Variation	Degree of Freedom (DF)	Sum of Squares (SS)	Mean Squares (MS)	F-Ratio	P-value
Factor A	$m-1$	$SSA = np \sum_{i=1}^{m} (\bar{Y}_{i\,..} - \bar{Y}_{...})^2$	$MSA = \frac{SSB}{m-1}$	$F_A = \frac{MSA}{MSE}$	P_A
Factor B	$n-1$	$SSB = mp \sum_{j=1}^{n} (\bar{Y}_{.j.} - \bar{Y}_{...})^2$	$MSB = \frac{SSB}{n-1}$	$F_B = \frac{MSB}{MSE}$	P_B
Interaction AB	$(m-1)(n-1)$	$SSAB = p \sum_{i=1}^{m} \sum_{j=1}^{n} (\bar{Y}_{ij.} - \bar{Y}_{i\,..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$	$MSAB = \frac{SSB}{(m-1)(n-1)}$	$F_{AB} = \frac{MSAB}{MSE}$	P_{AB}
Error	$mn(p-1)$	$SSE = SST - SSA - SSB - SSAB$	$MSE = \frac{SSE}{mn(p-1)}$		
TOTAL	$mnp-1$	$SST = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} (y_{ijk} - \bar{Y}_{...})^2$			

See Gupta (2011)

Two Way Anova Model With Two Random Effects

When models include random effects, the Expected Mean Squares will often differ from the same model with fixed effects. In most cases, this will affect how F-tests are performed, and the distribution of the F-statistic and its degrees of freedom, James (2013)

Table 3. Summary Table For ANOVA With Two Random Effects

Source of Variation	Degree of Freedom (DF)	Sum of Squares (SS)	Mean Squares (MS)	F-Ratio	P-value
Factor A	$m-1$	$SSA = np \sum_{i=1}^{m} (\bar{Y}_{i..} - \bar{Y}_{...})^2$	$MSA = \frac{SSB}{m-1}$	$F = \frac{MSA}{MSAB}$	P_A
Factor B	$n-1$	$SSB = mp \sum_{j=1}^{n} (\bar{Y}_{.j.} - \bar{Y}_{...})^2$	$MSB = \frac{SSB}{n-1}$	$F = \frac{MSB}{MSAB}$	P_B
Interaction AB	$(m-1)(n-1)$	$SSAB = p \sum_{i=1}^{m} \sum_{j=1}^{n} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$	$MSAB = \frac{SSB}{(m-1)(n-1)}$	$F = \frac{MSAB}{MSE}$	P_{AB}
Error	$mn(p-1)$	$SSE = SST - SSA - SSB - SSAB$	$MSE = \frac{SSE}{mn(p-1)}$		
TOTAL	$mnp - 1$	$SST = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} (y_{ijk} - \bar{Y}_{...})^2$			

See James (2013)

Two Way Anova Model With Fixed And Random Effects

The balanced two way ANOVA model with fixed and random effects is given by:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \quad i = 1,2,...,m; \quad j = 1,2,...,n \text{ and}$$

$$k = 1,2,...,p \quad ....(2.9)$$

where

$\varepsilon_{ijk} \sim iid \ N(0, \sigma_\alpha^2)$, Random factor B: $\beta_j \sim iid \ N(0, \sigma_\beta^2)$ and Fixed factor A

$$: \sum_{i=1}^{m} \alpha_i = 0 ...(2.10)$$

Interaction effect $(\alpha\beta)_{ij} \sim iid \ N\left(0, \frac{m-1}{m}\sigma_{\alpha\beta}^2\right)$ subject to the restriction

$$\begin{cases} \sum_{i=1}^{m} (\alpha\beta)_{ij} = 0 \\ Cov\left((\alpha\beta)_{ij,} (\alpha\beta)_{i^1 j}\right) = -\frac{1}{m}\sigma_{\alpha\beta}^2 \ \forall i \neq i^1 \end{cases} ....(2.11)$$

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and ε_{ijk}, β_j and $(\alpha\beta)_{ij}$ are pairwise independent

1. $E(Y_{ijk}) = \mu + \alpha_i$ ....(2.12)

2. $Var(Y_{ijk}) = \sigma^2 + \sigma_\beta^2 + \dfrac{m-1}{m}\sigma_{\alpha\beta}^2$ ....(2.13)

The covariance structure of the response is given by:

$$Cov(Y_{ijk}\,{}'Y_{i^1 jk^1}) = \sigma_\beta^2 - \frac{1}{m}\sigma_{\alpha\beta}^2 \;\forall i \neq i^1 \quad\text{...(2.14)}$$

$$Cov(Y_{ijk}\,{}'y_{ij^1k^1}) = \sigma_\alpha^2 \;\forall j \neq j^1 \quad\text{...(2.15)}$$

$$Cov(Y_{ijk}\,{}'y_{ij\,k^1}) = \sigma_\beta^2 + \frac{m-1}{m}\sigma_{\alpha\beta}^2 \;\forall k \neq k^1 \quad\text{...(2.16)}$$

$$Cov(Y_{ijk}\,{}'y_{i^1 j^1 k^1}) = 0;$$

$i \neq i^1, j \neq j^1, k \neq k^1$ ....(2.17)

Table 3. Summary Table For Two Way ANOVA With Fixed and Random Effects

Source of Variation	Degree of Freedom (DF)	Sum of Squares (SS)	Mean Squares (MS)	F-Ratio	P-value
Factor A	$m-1$	$SSA = np\sum_{i=1}^{m}(\bar{Y}_{i..}-\bar{Y}_{...})^2$	$MSA = \dfrac{SSB}{m-1}$	$F = \dfrac{MSA}{MSAB}$	P_A
Factor B	$n-1$	$SSB = mp\sum_{j=1}^{n}(\bar{Y}_{.j.}-\bar{Y}_{...})^2$	$MSB = \dfrac{SSB}{n-1}$	$F = \dfrac{MSB}{MSE}$	P_B
Interaction AB	$(m-1)(n-1)$	$SSAB = p\sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}+\bar{Y}_{...})^2$	$MSAB = \dfrac{SSB}{(m-1)(n-1)}$	$F = \dfrac{MSAB}{MSE}$	P_{AB}
Error	$mn(p-1)$	$SSE = SST - SSA - SSB - SSAB$	$MSE = \dfrac{SSE}{mn(p-1)}$		
TOTAL	$mnp-1$	$SST = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}(y_{ijk}-\bar{Y}_{...})^2$			

James (2013)

P-Value-Based Bayes Factor Proposed Held And Ott (2018)

The commonly used calibration $(-ep\log_e p)$, proposed by Sellke et al. (2001), depends directly on the p-value p, thus:

$$\text{min BF(p)} = \begin{cases} -e & p\log_e p \\ & 1 \end{cases}$$

$$\begin{aligned} &for \quad p < e^{-1} \\ &\quad otherwise \end{aligned} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2.18)$$

A simple derivation of Equation (2.18) assumes that under a point null hypothesis H_0, an exact $p-value, p$ is uniformly distributed on the unit interval. Under the alternative hypothesis, small p-values are expected, so the class of beta prior distributions $Be(\xi, 1)$ with monotonically decreasing density functions $(\xi \leq 1)$ is considered.

The minimum Bayes factor (Equation 2.18) can then be derived as described by Sellke et al. (2001) and Held &Ott (2016), using the maximum likelihood estimate

$$(MLE)\xi_{ML} = \min\left\{-\frac{1}{\log_e(p)}, 1\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2.19)$$

Sellke et al. (2001) also presented an alternative derivation of Equation (2.18), in which one does not have to assume the beta class for the p-value under the alternative hypothesis (H_1). Held (2016) noted that Equation (2.18) can also be derived as a test-based Bayes factor under the g-prior if θ has dimension 2 and the sample size is large. The calibration given by Equation (2.18) is always smaller than the local normal alternatives bound (See Held and Ott, 2018) and approximately equal to the lower bound in the more general class of all local alternatives, Sellke et al. (2001).

The beta distribution $Be(\xi, 1)$ with $(\xi \leq 1)$ has prior sample size $(\xi + 1 \leq 2)$, so it is always quite uninformative. Therefore, $f(p|H_1, \xi_{ML})$ will be relatively flat, and the minimum Bayes factor

$$\text{min} \qquad\qquad\qquad\qquad \text{BF(p)} \qquad\qquad\qquad\qquad\qquad =$$

$$\frac{1}{f(p/H_1\xi_{ML})} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(2.20)$$

will be relatively large. However, this is not the only class of beta priors with monotonically decreasing density functions. An alternative, which to our knowledge has not yet been discussed in the literature, that is class of beta distributions $Be(1, \kappa)$ with $(\kappa \geq 1)$.

A beta distribution from this class has prior sample size $(1 + \kappa \geq 2)$, so the likelihood under the alternative can take larger values than for the above $Be(\xi, 1)$ prior. Calculus shows that in this setting,

MLE of k is

$$\hat{K}_{ML} = \max\left\{-\frac{1}{\log_e(1-p)}, 1\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2.21)$$

leading to the minimum Bayes factor

$$\min BF(p) = \begin{cases} -e \ (1-p)\log_e(1-p) \\ 1 \end{cases}$$

for $p < 1 - e^{-1}$

otherwise   ………………………(2.22)

This is similar to the $-e\,p\log_e p$ calibration, but with p replaced by

$q = 1 - p$

………………………..…………………………………………………….(2.23)

so we call this the $-e\,q\,log\,q$ calibration. Note that for small enough p we can obtain the simple approximate formula $minBF(p) \approx e\,p$ based on the approximation $log(1 - p) \approx -p$.

It turns out that Equation (2.22) is a much lower bound compared with that proposed by Sellke*et al.* (2001). For *p*-values less than 0.1, it is even smaller than the Goodman approach(Goodman, 2016). This is due to a large (and unbounded) prior sample size for small p, in contrast to the prior sample size of the $-e\,p\,log\,p$ calibration, which cannot be larger than 2.

However, Equation (2.22) provides a sharp lower bound on Bayes factors based on *g*-priors of any dimension *d*, even if the sample size is very small. For reasonably large sample sizes, however, the $-e\,q\,log\,q$ calibration will be too conservative, (Held and Ott, 2018).

Table 4. Categorization of Bayes Factor$(BF) < 1$ into levels of evidence against

the null hypothesis (H_0)

Bayes Factor (BF_{01})	Strength of Evidence Against the null hypothesis (H_0)
$1 - \dfrac{1}{3}$	Weak
$\dfrac{1}{3} - \dfrac{1}{10}$	Moderate
$\dfrac{1}{10} - \dfrac{1}{30}$	Substantial
$\dfrac{1}{30} - \dfrac{1}{100}$	Strong
$\dfrac{1}{100} - \dfrac{1}{300}$	Very Strong
$< \dfrac{1}{300}$	Decisive

Held and Ott (2016)

The study was carried out in the following steps:
STEPS:
1. For each case (1, 2 and 3), data was simulated for the set, m**,** n and p combination.
2. The frequentist Two Way ANOVA table summary was computed using the simulated data for each case and the corresponding P-values obtained.
3. The Minimum Bayes factor proposed by Held and Ott (2018) was computed using results in 2 above.
4. The results was discussed

DATA PRESENTATION AND ANALYSIS
Data Presentation

Data sets were simulated using the native functions implemented in the R software for statistical computing (version 3.4.0 for Windows, R Core Team, 2017) from a standard normal population $N(\mu = 0, \sigma = 1)$. Simulation was generated using random seed sets to simplify replication. The random sample generated contains 125 random numbers clustered in 25 cells (5 rows and 5 columns). Each cell contains 5 random numbers. This design is similar to a setup of the regular two way ANOVA with five replicates per cell. **Table** 3.1 below was generated from the process above.

Table 5. Simulated Data`

		FACTOR B				
		Level 1	**Level 2**	**Level 3**	**Level 4**	**Level 5**
FACTOR A	**Level 1**	-0.90, 0.18, 1.59, -1.13, -0.08	0.13, 0.71, -0.24, 1.98, -0.14	0.42, 0.98, -0.39, -1.04, 1.78	-2.31, 0.88, 0.04, 1.01, 0.43	2.09, -1.20, 1.59, 1.95, 0.00
	Level 2	-2.45, 0.48, -0.60, 0.79, 0.29	0.74, 0.32, 1.08, -0.28, -0.78	-0.60, -1.73, -0.90, -0.56, -0.25	-0.38, -1.96, -0.84, 1.90, 0.62	1.99, -0.31, -0.09, -0.18, -1.20
	Level 3	-0.84, 2.07, -0.56, 1.28, -1.05	-1.97, -0.32, 0.94, 1.14, 1.67	-1.79, 2.03, -0.70, 0.16, 0.51	-0.82, -2.00, -0.48, -0.08, -0.90	-0.92, 0.33, -0.14, 0.43, -0.05
	Level 4	-0.91, 1.30, 0.77, 1.05, -1.41	1.00, -1.70, -0.53, -1.37, -2.21	1.82, -0.65, -0.28, -0.39, 0.39	1.60, 1.68, -1.18, -1.36, -1.51	-1.25, 1.96, 0.01, -0.84, -0.86
	Level 5	1.07, 0.26, -0.31, -0.75, -0.35	2.05, 0.94, 2.01, -0.42, 0.56	-1.03, -0.25, 0.47, 1.36, -0.78	0.46, 1.23, 1.15, 0.11, -1.04	1.24, 0.14, 1.71, -0.43, -0.86

Source: Simulation Result

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From table 5 above,

$Number\ of\ rows\ (m) = 5$

$Number\ of\ Columns\ (n) = 5$

$Number\ of\ Observations\ per\ cell\ (p) = 5$

$$SST = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}(y_{ijk} - \bar{Y}_{...})^2 = \sum_{i=1}^{5}\sum_{j=1}^{5}\sum_{k=1}^{5}(y_{ijk} - \bar{Y}_{...})^2 = 158.0611$$

$$SSA = np\sum_{i=1}^{m}(\bar{Y}_{i..} - \bar{Y}_{...})^2 = 5(5)\sum_{i=1}^{5}(\bar{Y}_{i..} - \bar{Y}_{...})^2 = 7.4092$$

$$SSB = mp\sum_{j=1}^{n}(\bar{Y}_{.j.} - \bar{Y}_{...})^2 = 5(5)\sum_{j=1}^{5}(\bar{Y}_{.j.} - \bar{Y}_{...})^2 = 2.0779$$

$$SSAB = p\sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}) = 5\sum_{i=1}^{5}\sum_{j=1}^{5}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}) = 14.5775$$

$$SSE = SST - SSA - SSB - SSAB = 158.0611 - 7.4092 - 7.4092 - 2.0779 - 14.5775 = 133.9965$$

$$MSA = \frac{SSB}{m-1} = 1.8523$$

$$MSB = \frac{SSB}{n-1} = 0.5195$$

$$MSAB = \frac{SSB}{(m-1)(n-1)} = 0.9111$$

$$MSE = \frac{SSE}{mn(p-1)} = 1.3400$$

DATA ANALYSIS

Case 1: Both factors (i.e. factors A and B) are fixed

Table 6. ANOVA Summary Table for Case 1

Source of Variation	Degree of Freedom (DF)	Sum of Squares (SS)	Mean Squares (MS)	F-Ratio	P-Values
Factor A	$5-1=4$	$SSA = 7.4092$	$MSA = \dfrac{SSB}{m-1} = 1.8523$	$F_1 = \dfrac{MSA}{MSE} = 1.3823$	0.2455
Factor B	$5-1=4$	$SSB = 2.0779$	$MSB = \dfrac{SSB}{n-1} = 0.5195$	$F_2 = \dfrac{MSB}{MSE} = 0.3877$	0.8170
Interaction AB	$(5-1)(5-1)=16$	$SSAB = 14.5775$	$MSAB = \dfrac{SSB}{(m-1)(n-1)} = 0.9111$	$F_3 = \dfrac{MSAB}{MSE} = 0.6799$	0.8074
Error	$5(5)(5-1)=100$	$SSE = 133.9965$	$MSE = \dfrac{SSE}{mn(p-1)} = 1.3400$		
TOTAL	$5(5)(5)-1=124$	$SST = 158.0611$			

Hypothesis Testing for Case 1: Both factors (i.e. Factors A and B) are fixed

To test the hypothesis of interest, we obtained the minimum Bayes factor for each of the hypothesis using the p-values already computed in the table above:

$$H_{01} : \sigma_\alpha^2 = \text{against}$$

$$H_{11} : \sigma_\alpha^2 \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3.1)$$

The Held and Ott(2018) minimum Bayes factor is given by:

$$\min \text{BF}(p) = \begin{cases} -e \; (1-p)\log_e(1-p) & for \quad p < 1-e^{-1} \\ 1 & otherwise \end{cases}$$ See equation

(2.22)

Since Pvalue = 0.2545 which is less than p = 1- e^{-1} = 1 – (2.718)$^{-1}$ = 0.6321

$$minBF(p) = -e\,(1-p)\log_e(1-p) = -2.718(1-0.2455)\log_e(1-0.2455)$$
$$= -2.718(0.7545)\log_e(0.7545) = 0.5777$$

Table 7. Categorization of Bayes Factor $(BF) < 1$ into levels of evidence against the null hypothesis (H_0)

Bayes Factor (BF_{01})	Strength of Evidence Against the null hypothesis (H_0)
$1 - \dfrac{1}{3}$	Weak
$\dfrac{1}{3} - \dfrac{1}{10}$	Moderate
$\dfrac{1}{10} - \dfrac{1}{30}$	Substantial
$\dfrac{1}{30} - \dfrac{1}{100}$	Strong
$\dfrac{1}{100} - \dfrac{1}{300}$	Very Strong
$< \dfrac{1}{300}$	Decisive

Held and Ott (2016)

The Bayes factor $BF_{10} = 0.5777$, signifies that the data has a weak evidence against the null hypothesis of no variability in the five levels of factor A stated in equation (3.1). This can be seen in **Table 3.3.**

To test the hypothesis below,

$$H_{02} : \sigma_\beta^2 = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{against}$$

$$H_{12} : \sigma_\beta^2 \neq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.2)$$

Held and Ott (2018) minimum Bayes factor is given by:

$$\text{min BF(p)} = \begin{cases} -e \ (1-p)\log_e(1-p) & for \quad p < 1 - e^{-1} \\ 1 & otherwise \end{cases} \quad \text{See equation}$$

(2.22)

Since Pvalue = 0.8170 is greater than p = 1- e^{-1} = 1 – (2.718)$^{-1}$ = 0.6321

Hence, $minBF(p) = 1$

The Bayes factor $BF_{10} = 1$, signifies that the data has a weak evidence against the null hypothesis of no variability in the five levels of factor B stated in equation (3.2). This can be seen in **Table 7.**

To test the hypothesis below

$$H_{03}: \sigma^2_{\alpha\beta} = 0 \qquad\qquad \text{against}$$

$$H_{13}: \sigma^2_{\alpha\beta} \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3.3)$$

Held and Ott (2018) minimum Bayes factor is given by:

$$\min BF(p) = \begin{cases} -e \quad (1-p)\log_e(1-p) & for \quad p < 1-e^{-1} \\ 1 & otherwise \end{cases}$$ See equation

(2.22)

Since Pvalue = 0.8170 is greater than p = 1- e^{-1} = 1 – (2.718)$^{-1}$ = 0.6321, min BF(p) = 1

The Bayes factor $BF_{10} = 1$, signifies that the data has a weak evidence against the null hypothesis of no variability in the interaction effects of factors A and B stated in equation (3.3). This can be seen in **Table 7.**

DISCUSSION FOR CASE 1 (BOTH FACTORS FIXED)

In all the three hypothesis examined above, the data indicated a weak evidence against the null hypothesis stated in equations 3.1, 3.2 and 3.3. This shows that the data strongly supports the null hypothesis of no variability in the effects of the levels of the factors and in the interaction effects. This result corresponds to the frequentist conclusion based on the p-values; in that all the p-values are greater than 0.05 significance levels. This shows that for a two model with fixed effects, the Bayesian as well as the frequentist conclusions are not differing.

Case 2: Both Factors Are Random (i.e. two random effects)

Table 8. ANOVA Summary Table for Case 2

Source of Variation	Degree of Freedom (DF)	Sum of Squares (SS)	Mean Squares (MS)	F-Ratio	P-Values
Factor A	$5-1=4$	$SSA = 7.4092$	$MSA = \dfrac{SSB}{m-1} = 1.8523$	$F_1 = \dfrac{MSA}{MSAB} = 2.0330$	0.1380
Factor B	$5-1=4$	$SSB = 2.0779$	$MSB = \dfrac{SSB}{n-1} = 0.5195$	$F_2 = \dfrac{MSB}{MSAB} = 0.5702$	0.6881
Interaction AB	(5-1)(5-1)=16	$SSAB = 14.5775$	$MSAB = \dfrac{SSB}{(m-1)(n-1)} = 0.9111$	$\dfrac{MSAB}{MSE} = 0.6799$	0.8074
Error	$5(5)(5-1) = 100$	$SSE = 133.9965$	$MSE = \dfrac{SSE}{mn(p-1)} = 1.3400$		
TOTAL	$5(5)(5)-1 = 124$	$SST = 158.0611$			

8.Hypothesis Testing for Case 2: Both factors (i.e. Factors A and B) are Random

To test the hypothesis of interest, we obtain the Held and Ott(2018) minimumBayes factor for each of the hypothesis using the $p-values$ already computed in the table above:

$H_{01} : \sigma_\alpha^2 =$ against

$H_{11} : \sigma_\alpha^2 \neq 0$ ....(3.4)

Held and Ott (2018) minimum Bayes factor is given by:

$$\text{min BF(p)} = \begin{cases} -e \ (1-p)\log_e(1-p) & for \quad p < 1-e^{-1} \\ 1 & otherwise \end{cases}$$ See equation

(2.22)

Since Pvalue = 0.1380 which is greater than p = 1- e^{-1} = 1 – (2.718)$^{-1}$ = 0.6321, min

BF(p) = 1

$$minBF(p) = -e \ (1-p)\log_e(1-p) = -2.718(1-0.1380)\log_e(1-0.1380)$$
$$= -2.718(0.8620)\log_e(0.8620) = 0.3480$$

Table 9. Categorization of Bayes Factor $(BF) < 1$ into levels of evidence against the null hypothesis (H_0)

Bayes Factor (BF_{01})	Strength of Evidence Against the null hypothesis (H_0)
$1 - \dfrac{1}{3}$	Weak
$\dfrac{1}{3} - \dfrac{1}{10}$	Moderate
$\dfrac{1}{10} - \dfrac{1}{30}$	Substantial
$\dfrac{1}{30} - \dfrac{1}{100}$	Strong
$\dfrac{1}{100} - \dfrac{1}{300}$	Very Strong
$< \dfrac{1}{300}$	Decisive

Held and Ott (2016)

The Bayes factor $BF_{10} = 0.3480$, signifies that the data has a weak evidence against the null hypothesis of no variability in the five levels of factor A stated in equation (3.4). This can be seen in **Table 3.5.**
To test the two (2) hypotheses below,

$$H_{02} : \sigma_{\beta}^2 = 0 \qquad \qquad \text{against}$$

$$H_{12} : \sigma_{\beta}^2 \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3.5)$$

$$H_{03} : \sigma_{\alpha\beta}^2 = 0 \qquad \qquad \text{against}$$

$$H_{13} : \sigma_{\alpha\beta}^2 \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(3.6)$$

The Held and Ott(2018) minimum Bayes factor is given by:

$$\text{min BF(p)} = \begin{cases} -e \quad (1-p)\log_e(1-p) & \text{for} \quad p < 1-e^{-1} \\ 1 & \text{otherwise} \end{cases} \quad \text{See equation}$$

(2.22)

we can see from **Table 3.4** that the respective Pvalue (0.6881 and 0.8074) are all greater than $p = 1 - e^{-1} = 1 - (2.718)^{-1} = 0.6321$. Hence, min BF(p) = 1 for both hypotheses.

The Bayes factor $BF_{10} = 1$, signifies that the data has a weak evidence against the null hypothesis of no variability in levels of factor B (stated in equation (3.5)) and also the interaction effects of factors A and B stated in equation (3.6). This can be seen in **Table 3.5** above.

DISCUSSION FOR CASE 2 (Two Random Effects)

In testing the three (3) hypotheses stated in equations (3.4), (3.5) and (3.6), we realized that the data shows a weak evidence against of the null hypothesis of no variability in the effects of the levels of the factors A and B as well as the effects their interactions. A side examination of a frequentist conclusion from the p-values at a 5% significance level,indicates that the null hypothesis of zero treatment effect stated in equations (3.4), (3.5) and (3.6) ought not to be rejected. Hence, the Bayesian as well as the frequentist conclusions are not differing.

3.2.7   Case 3: Factor A is Fixed and Factor B is Random (i.e. typical of a mixed effect model)

Table 10. ANOVA Summary Table for Case 3

Source of Variation	Degree of Freedom (DF)	Sum of Squares (SS)	Mean Squares (MS)	F-Ratio	P-Value
Factor A	$5 - 1 = 4$	$SSA = 7.4092$	$MSA = \dfrac{SSB}{m-1} = 1.8523$	$F_A = \dfrac{MSA}{MSAB} = 2.0330$	0.1380
Factor B	$5 - 1 = 4$	$SSB = 2.0779$	$MSB = \dfrac{SSB}{n-1} = 0.5195$	$F_B = \dfrac{MSB}{MSE} = 0.3877$	0.8170
Interaction AB	$(5\text{-}1)(5\text{-}1) = 16$	$SSAB = 14.5775$	$MSAB = \dfrac{SSB}{(m-1)(n-1)} = 0.9114$	$\dfrac{MSAB}{MSE} = 0.6799$	0.8074
Error	$5(5)(5-1) = 100$	$SSE = 133.9965$	$MSE = \dfrac{SSE}{mn(p-1)} = 1.3400$		
TOTAL	$5(5)(5) - 1 = 124$	$SST = 158.0611$			

10. Hypothesis Testing for Case 2: Both factors (i.e. Factors A and B) are Random

To test the hypothesis of interest, we obtain the Held and Ott (2018) minimum Bayes factor for each of the hypothesis using the p-values already computed in the **Table 3.6**above:

$$H_{01} : \sigma_\alpha^2 = \qquad \text{against}$$

$$H_{11} : \sigma_\alpha^2 \neq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.7)$$

Held and Ott(2018) minimum Bayes factor is given by:

$$\min BF(p) = \begin{cases} -e \ (1-p)\log_e(1-p) & \text{for} \quad p < 1-e^{-1} \\ 1 & \text{otherwise} \end{cases}$$ See equation

(2.22)

Since Pvalue = 0.1380 which is greater than $p = 1 - e^{-1} = 1 - (2.718)^{-1} = 0.6321$

$$minBF(p) = -e\,(1-p)\log_e(1-p) = -2.718(1 - 0.1380)\log_e(1 - 0.1380)$$
$$= -2.718(0.8620)\log_e(0.8620) = 0.3480$$

Table 11. Categorization of Bayes Factor $(BF) < 1$ into levels of evidence against the null hypothesis (H_0)

Bayes Factor (BF_{01})	Strength of Evidence Against the null hypothesis (H_0)
$1 - \dfrac{1}{3}$	Weak
$\dfrac{1}{3} - \dfrac{1}{10}$	Moderate
$\dfrac{1}{10} - \dfrac{1}{30}$	Substantial
$\dfrac{1}{30} - \dfrac{1}{100}$	Strong
$\dfrac{1}{100} - \dfrac{1}{300}$	Very Strong
$< \dfrac{1}{300}$	Decisive

Held and Ott (2016)

The Bayes factor $BF_{10} = 0.3480$, signifies that the data has a weak evidence against the null hypothesis of no variability in the five levels of factor A stated in equation (3.7). This can be seen in **Table 3.7.**
To test the two (2) hypotheses below,

$$H_{02} : \sigma_\beta^2 = 0 \qquad \text{against}$$

$$H_{12} : \sigma_\beta^2 \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3.8)$$

$$H_{03} : \sigma_{\alpha\beta}^2 = 0 \qquad \text{against}$$

$$H_{13} : \sigma_{\alpha\beta}^2 \neq 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3.9)$$

Held and Ott (2018) minimum Bayes factor is given by:

$$\min BF(p) = \begin{cases} -e \ (1-p)\log_e(1-p) & for \quad p < 1 - e^{-1} \\ 1 & otherwise \end{cases}$$

See equation

(2.22)

we can see from Table 11. that the respective Pvalues (0.8170 and 0.8074) are all greater than

$p = 1 - e^{-1} = 1 - (2.718)^{-1} = 0.6321$. Hence, $\min BF(p) = 1$ for both hypotheses.

The Bayes factor $BF_{10} = 1$, signifies that the data has a weak evidence against the null hypothesis of no variability in levels of factor B (stated in equation (3.8)) and also the interaction effects of factors A and B stated in equation (3.9). This can be seen in **Table 3.7**above**.**

DISCUSSION FOR CASE 3 (Factor A is Fixed and Factor B is Random)

In testing the three (3) hypotheses stated in equations (3.7), (3.8) and (3.9), we realized that the data shows a weak evidence against of the null hypothesis of no variability in the effects of the levels of the factors A and B as well as the effects of their interactions. A side examination of a frequentist conclusion from the p-values at a 5% significance level, indicates that the null hypothesis of zero treatment effect stated in equations (3.7), (3.8) and (3.9) will not to be rejected. Hence, the Bayesian as well as the frequentist conclusions are not differing.

CONCLUSIONS AND RECOMMENDATIONS

In all the three cases studied in this paper, it is evident that the Held and Ott (2018) minimum Bayes factor is conservative. In fact, at no point did the data exhibit strong evidence against the null hypothesized model of zero variability. We realized that the calibrated p-value based minimum Bayes factors do not differ in conclusions from those of the frequentist. When the same data was subjected to the Faulkenberry (2018) Bayes factor, the conclusions for the fixed effects differed significantly from those of the random effects (**See** Egburonu and Abidoye, 2018); but the conservative nature of the minimum Bayes factor overcame this difference. The minimum Bayes factor provides lesser values for Bayes factor when compared to the Prior sensitive and BIC based Bayes factors proposed by Wang and Sun (2011) and Faulkenberry (2018) respectively.

FURTHER STUDY

This research still has limitations so that further research is still needed on this topic.

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