



An Effective Method of Graceful Labeling for Pendant Graphs

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ABSTRACT

This study focuses on the significant branch of graph theory known as graceful labeling, which involves assigning integers to the vertices and edges of graphs. Various techniques, such as vertex-graceful, edge-graceful, harmonious, lucky, magic, and prime labeling, have been developed to address this problem. Despite the extensive research on graceful labeling, the specific challenge of labeling pendant graphs gracefully has not been widely explored. Our research proposes new algorithms for gracefully labeling graphs with pendant vertices. These algorithms can be applied to various types of graphs, including cyclic, tetrahedron, regular, octahedron, complete, and square pyramid graphs. By introducing these new methods, we aim to fill the gap in the literature regarding pendant graphs. The study concludes with a detailed case study that illustrates the practical application of the proposed algorithms, demonstrating their effectiveness and ease of use in gracefully labeling pendant graphs. This contribution provides a valuable addition to the existing body of knowledge on graph labeling

INTRODUCTION

Graph theory is a captivating area of mathematics that offers essential insights into the structure and dynamics of complex networks. Among the numerous concepts and techniques that have emerged from this field, the study of graceful labeling stands out as a subject of particular elegance and intrigue. It involves assigning labels to the vertices of a graph such that specific conditions are met. One specific type of graph that has received considerable attention in the context of graceful labeling is graphs with pendant vertices. It has mainly two types of labeling called vertex labeling and edge labeling. For any given graph $G \equiv (V, E)$ it can be defined vertex labeling as a function of V to a set of labels and graph with such a function called "a vertex-labeled graph". That is the vertices of the graph are labeled with distinct positive integers in such a way that no two distinct vertices receive the same label, and the induced edge labels (absolute differences between incident vertex labels) are distinct (Santhakumaran and Balaganesan 2018). Edge labeling is a function of E to a set of labels and such a graph is called an "edge-labeled graph". That is the edges of the graph are labeled with distinct positive integers in such a way that no two distinct edges receive the same label, and the induced vertex labels (sums of incident edge labels) are distinct (Gayathri and Subbiah 2011).

The concept of graceful labeling was introduced by Alexander Rosa in 1967. (Aziz et al. 2015). After Rosa's initial work, researchers began exploring the properties of graphs that allow for graceful labeling and investigating specific classes of graphs. Some results include the determination of graceful trees and cycles. In 1978, mathematician Joseph A. Gallian proposed a conjecture related to graceful labeling, which stated that all trees are graceful. The conjecture remained open for a long time, and proving it for all trees became a significant challenge. In graph theory, a tree is a kind of connected, undirected graph in which any two vertices are connected by exactly one path (Gnang 2022). But up until now, it has not been confirmed whether it is correct or incorrect because no one has come up with suitable proof for that. Even though most of the results claim that some of the graphs are not graceful, most of the graphs are not like that. A graph that does not contain graceful labeling is called an "ungraceful graph or non-graceful graph" (Redl 2016). By using graceful labeling, it is possible to assign integer numbers to the vertices in any graph with pendant vertices. Pendant vertices are vertices that have exactly one degree (Sebastian and Kureethara 2018). Graceful labeling in the context of pendent graphs has several implications and applications. It is contributing to both theoretical aspects of graph theory and practical considerations in various fields. Labeled graphs play a crucial role as versatile models in various applications, including but not limited to coding theory, x-ray crystallography, radar systems, astronomy, circuit design, communication networks, database management, secret sharing schemes, and models for constraint programming over finite domains (Thejeshwi and Kirupa 2018).

In general, the study of graceful labeling in pendent graphs involves investigating whether it is possible to assign distinct positive integer labels to the edges in such a way that certain conditions, such as uniqueness of induced

vertex labels, are satisfied. The exploration of graceful labeling in pendent graphs contributes to the broader understanding of combinatorial properties and structural features within graph theory. The existence or non-existence of graceful labeling for specific classes of pendent graphs depends on their inherent properties and connectivity structures. Many graceful labeling algorithms have been developed for the different pendant graphs. For an instance by adding pendant vertices for these cycle vertices, it can create a cyclic graph with pendant vertices. A cyclic graph contains only one cycle. In that, all the vertices of a graph are connected by a closed chain. This research will introduce a graceful labeling algorithms for C_3 graphs (cyclic graphs with three vertices), tetrahedron graphs, and K_4 graph with pendant vertices.

LITERATURE REVIEW

Over the years, mathematicians and graph theorists continued to explore different types of graphs and conditions under which graceful labeling are possible. There have been studies on paths, cycles, complete graphs, and various other graph families. Therefore, research has focused on graceful labeling for specific classes of graphs, such as pendent graphs and trees. The study of graceful labeling in these contexts involves exploring the existence, properties, and constructions of graceful labeling. The field of graph labeling, introduced in the mid-1960s, has seen the exploration of approximately 200 techniques across more than 3000 research papers. It's noteworthy that the majority of graphs do not possess graceful labeling. However, graphs tend to exhibit gracefulness when there is a degree of regularity or structure inherent in their composition. This observation underscores the significance of structural regularity in facilitating the establishment of graceful labeling for graphs (Gallian 2018; Graham and Sloane 1980). Since 1975, there has been recognition that rooted symmetric trees possess the quality of gracefulness (Hrnčiar and Haviar 2001). An algorithm was proposed to generate a graceful tree from any arbitrary tree. This algorithm achieves this by introducing a sequence of new pendent edges to the initial arbitrary tree, establishing the significant result that every tree can be regarded as a subtree within a graceful tree (Sethuraman and Ragukumar 2017). During the early 1980s, the concept of graceful labeling expanded to include directed graphs. This extension involved the definition of graceful labeling on a directed graph $D(V, E)$ as a one-to-one mapping (Shivarajkumar, Sriraj, and Hegde 2021; Annamalai and Adhimoolam 2022; Beutner and Harborth 2002). The findings include outcomes of graceful labeling for various directed structures such as paths, stars, wheels, and umbrellas. The provided evidence supports the assertion that graceful labeling can be achieved for the disjoint union of three instances of a directed cycle, each having a predetermined even length (Jinsiqintuya 2012).

Preliminaries

A graph consists of vertices and edges. Each vertex is denoted by a point. An edge joins two vertices, which are often called the endpoints of the edge. We usually depict an edge as a straight-line segment joining the two endpoints. A graph represents the relationship between an edge and a pair of vertices. Normally, a graph must have at least one vertex, but it need not have any edges. Let us consider some fundamental definitions.

Definition 01: Graph

A simple graph G is defined as a pair $[V(G), E(G)]$, such that $V(G)$ is a non-empty finite set of elements called “vertices” and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$ called “edges” (Bell, 2015)

Definition 02: Cycle Graph

The cycle graph contains a single cycle. That is a closed chain. In the cycle graph, every vertex has degree 2. For “ n ” vertices ($n \geq 3$), the cycle on “ n ” vertices, is denoted by C_n . The cycles are normally drawn to resemble geometric figures: C_3 as a triangle, C_4 as a square, C_5 as a pentagon, and so on (Mathew, 2014).

Definition 03: Tetrahedral Graph

The tetrahedral graph holds a unique position among Platonic graphs. It is the only polyhedral graph with four nodes, and it also belongs to both the complete graph and wheel graph families.

Definition 04: Complete Graph

In a graph, if any two vertices are adjacent then the graph is called a complete graph. For “ n ” vertices complete graph has $\frac{n(n-1)}{2}$ number of edges. The complete graph is denoted as K_n for “ n ” vertices (Algorithms, 2023).

Definition 05: Directed and Undirected Graphs

Let's consider a graph $G \equiv [V(G), E(G)]$ consisting of vertices and edges. It is called a “directed graph” if the edge set is composed of ordered vertex pairs. A graph is called an “undirected graph” if the edge set is composed of unordered vertex pairs (Samanta et al., 2021).

Definition 06: Degree of Vertex

In an undirected graph the total number of edges that intersect with the vertex called as degree of the vertex. In a loop, the vertex contributes twice to the degree of that vertex. The degree of vertex V is denoted by $\deg(V)$ or $\delta(V)$ (Dharmendra & Kumar, 2013).

Definition 07: Pendant Vertices and Pendant Edges

A Pendant vertex is a vertex if and only if it has degree one. Similarly, it can be defined pendant edge is an edge if and only if one of its vertices is a pendant vertex (Graf, 2014a).

Definition 08: Graceful Labeling

Let $G \equiv (V, E)$ be a graph where $V(G)$ = set of vertices and $E(G)$ = set of edges in graph G , with the number of edges = $|E(G)| = m$. A graceful labeling of a graph G is a vertex labeling $f: V \rightarrow [0, m]$ such that f is injective and the edge labeling $f_e \rightarrow (UV) = |f(U) - f(V)|$ is also injective. A simple connected

graph with m edges is said to be graceful if its vertices and edges are labeled such that:

- 1) Each vertex is assigned a distinct integer from $\{0, \dots, m\}$,
- 2) Each edge is assigned a distinct integer from the set $\{0, \dots, m\}$ such that the label is equal to the absolute value of the difference between the two vertices the edge connects (Graf, 2014b).

The graphs that admit graceful labeling are called "Graceful graphs." (Uma & Murugesan, 2014)

METHODS

In this section, a new algorithm for vertex graceful labeling in a cycle graph that has three cycle vertices (C_3) with pendant vertices will be proposed.

Step 01: Denote the cycle vertices U_i as $\{U_1, U_2, U_3\}$, where $i = 1, 2, 3$.

Step 02: Determine the total number of pendant vertices that we want to distribute throughout the cycle vertices. (Say "P")

Step 3: Distribute the total pendant vertices among each cycle vertices symmetrically.

Step 4: Denote the pendant vertices that are adjacent to the $U_1, U_2,$ and U_3 cycle vertices as $\{U_{1j}, U_{2j},$ and $U_{3j}\}$, respectively. Where $j = 1, 2, 3, \dots, \frac{P}{3}$

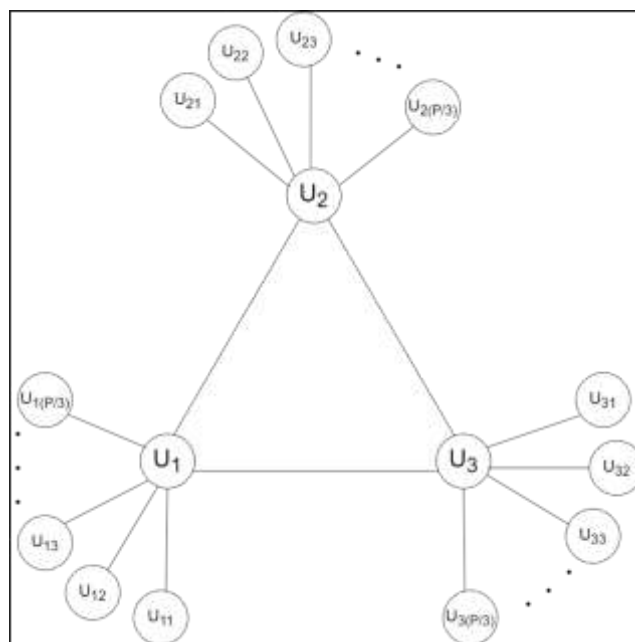


Figure 1. A Cyclic Graph (C_3) with "P" Number of Pendant Vertices

Step 05: Use the following functions $f(U_i)$ and $f(U_{ij})$ to assign the integer values for cycle vertices and pendant vertices respectively.

Formula 01

$$f(U_i) = \begin{cases} U_1 = 0 \\ U_2 = P+3 \end{cases}$$

$$U_3 = 1$$

$$f(U_{ij}) = \begin{cases} U_{1j} = U_1 + j + 1 & ; j = 1, 2, \dots, \frac{P}{3} \\ U_{2j} = \frac{U_2}{3} + j & ; j = 1, 2, \dots, \frac{P}{3} \\ U_{3j} = U_2 \frac{P}{3} + j + 1 & ; j = 1, 2, \dots, \frac{P}{3} \end{cases}$$

Step 06: Calculate the $|f(U_i) - f(U_{ij})|$ values and assign those integer values for each edge.

Step 07: Verify whether is each vertex assigned an integer from $\{0, 1, \dots, m\}$ and is each edge assigned a distinct integer from the set $\{0, \dots, m\}$ such that the label is equal to the absolute value of the difference of the two vertices the edge connects.

Now let's try to introduce a new algorithm to show that a tetrahedron graph with pendant vertices is graceful. The tetrahedron graph is a complete graph with four vertices, and the edges represent the edges of a tetrahedron.

Step 01: Denote the cycle vertices of tetrahedron U_i as $\{U_1, U_2, U_3\}$, where $i = 1, 2, 3, 4$ and the center vertex as U_4 .

Step 02: Determine the total number of pendant vertices that we want to distribute throughout the vertices at the base of the tetrahedron (U_1, U_2, U_3). (Say "P")

Step 03: Distribute the total pendant vertices among $\{U_1, U_2, U_3\}$ vertices symmetrically.

Step 04: Denote the pendant vertices which are adjacent with $U_1, U_2,$ and U_3 cycle vertices as $\{U_{1j}, U_{2j}, U_{3j}\}$ respectively. Where $j = 1, 2, \dots, \frac{P}{3}$

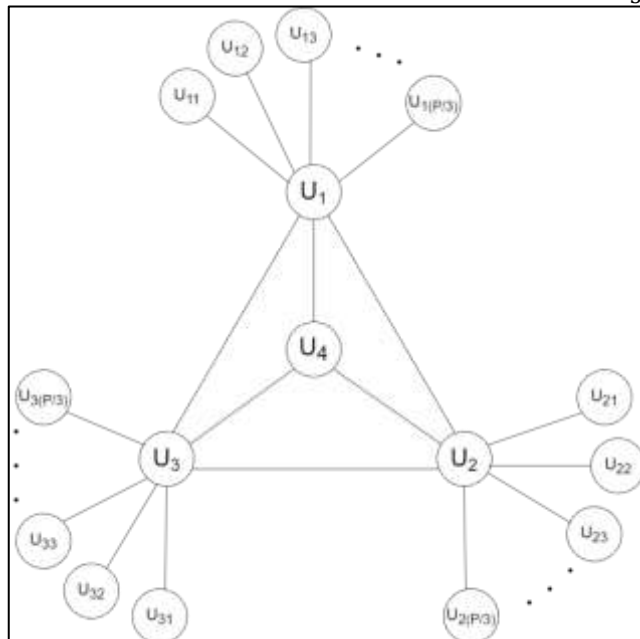


Figure 2. Tetrahedron Graph with "P" Number of Pendant Vertices

Step 05: Use the following functions $f(U_i)$ and $f(U_{ij})$ to assign the integer values for vertices in tetrahedron and pendent vertices which are adjacent to vertices in base respectively.

Formula 02

$$\begin{aligned}
 f(U_i) &= \left\{ \begin{array}{l} U_1 = 0 \\ U_2 = P + 5 \\ U_3 = 2 \\ U_4 = P + 6 \end{array} \right. \\
 f(U_{ij}) &= \left\{ \begin{array}{l} U_{1j} = U_1 + j + 2 \quad ; j = 1, 2, \dots, \frac{P}{3} \\ U_{2j} = U_1 \frac{P}{3} + j \quad ; j = 1, 2, \dots, \frac{P}{3} \\ U_{3j} = U_2 \frac{P}{3} + j + 2 ; j = 1, 2, \dots, \frac{P}{3} \end{array} \right.
 \end{aligned}$$

Step 06: Calculate the $|f(U_i) - f(U_{ij})|$ values and assign those integer values for each edge.

Step 07: Verify whether is each vertex assigned an integer from $\{0, 1, \dots, m\}$ and is each edge assigned a distinct integer from the set $\{0, \dots, m\}$ such that the label is equal to the absolute value of the difference of the two vertices the edge connects.

Consider the case to show a regular graph with pendant vertices is graceful. In regular graph degree of each vertex is equal. In the graph we are considering, each vertex has a degree of three.

Step 01: Denote the vertices of regular graph U_i as $\{U_1, U_2, U_3, U_4, U_5, U_6\}$, where $i = 1, 2, 3, 4, 5, 6$

Step 02: Determine the total number of pendant vertices that we want to distribute throughout the vertices at the base of the regular graph (U_1, U_2, U_3) . (Say "P")

Step 03: Distribute the total pendant vertices among $\{U_1, U_2, U_3\}$ vertices symmetrically.

Step 04: Denote the pendant vertices which are adjacent with $U_1, U_2,$ and U_3 cycle vertices as $\{U_{1j}, U_{2j}, U_{3j}\}$ respectively. Where $j = 1, 2, \dots, \frac{P}{3}$

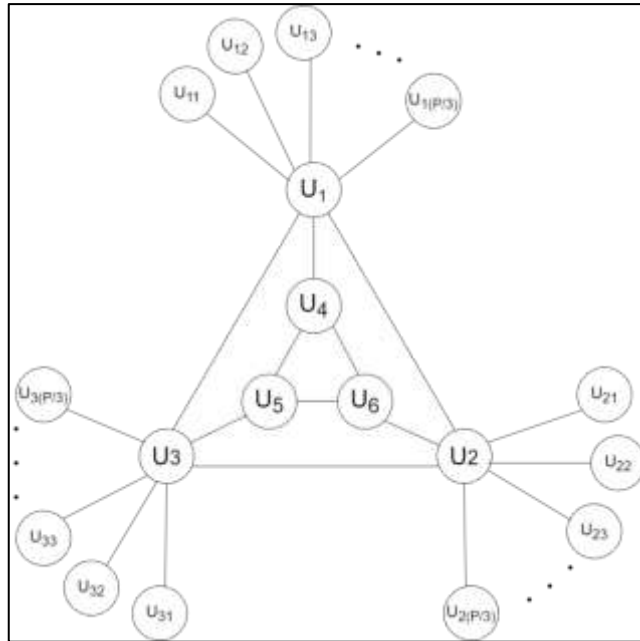


Figure 3. A regular Graph with “P” Number of Pendant Vertices

Step 05: Use the following functions $f(U_i)$ and $f(U_{ij})$ to assign the integer values for vertices in tetrahedron and pendent vertices which are adjacent to vertices in base respectively.

Formula 03

$$\begin{aligned}
 f(U_i) = & \left\{ \begin{array}{l} U_1 = P + 7 \\ U_2 = 5 \\ U_3 = 2 \\ U_4 = 0 \\ U_5 = P + 8 \\ U_6 = P + 9 \end{array} \right. \\
 f(U_{ij}) = & \left\{ \begin{array}{l} U_{1j} = 3 + 3j \quad ; j = 1, 2, \dots, \frac{P}{3} \\ U_{2j} = U_{1j} + 1 \quad ; j = 1, 2, \dots, \frac{P}{3} \\ U_{3j} = U_{1j} + 2 \quad ; j = 1, 2, \dots, \frac{P}{3} \end{array} \right.
 \end{aligned}$$

Step 06: Calculate the $|f(U_i) - f(U_{ij})|$ values and assign those integer values for each edge.

Step 07: Verify whether is each vertex assigned an integer from $\{0, 1, \dots, m\}$ and is each edge assigned a distinct integer from the set $\{0, \dots, m\}$ such that the

label is equal to the absolute value of the difference of the two vertices the edge connects.

Consider an octahedron graph with pendant vertices. The octahedron graph has 6 nodes and 12 edges.

Step 01: Denote the vertices of octahedron graph U_i as $\{U_1, U_2, U_3, U_4, U_5, U_6\}$, where $i = 1, 2, 3, 4, 5, 6$

Step 02: Determine the total number of pendant vertices that we want to distribute throughout the vertices at the base of the octahedron graph (U_1, U_2, U_3). (Say "P")

Step 03: Distribute the total pendant vertices among $\{U_1, U_2, U_3\}$ vertices symmetrically.

Step 04: Denote the pendant vertices which are adjacent with U_1, U_2 , and U_3 cycle vertices as $\{U_{1j}, U_{2j}, U_{3j}\}$ respectively. Where $j = 1, 2, \dots, \frac{P}{3}$

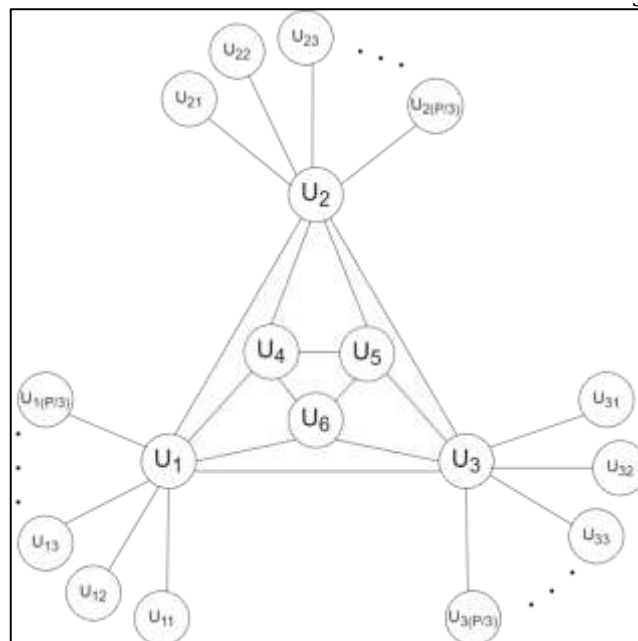


Figure 4. An octahedron Graph with "P" Number of Pendant Vertices

Step 05: Use the following functions $f(U_i)$ and $f(U_{ij})$ to assign the integer values for vertices in octahedron graph and pendant vertices which are adjacent to vertices in base respectively.

Formula 04

$$f(U_i) = \begin{cases} U_1 = P + 6 \\ U_2 = P + 11 \\ U_3 = 3 \\ U_4 = 0 \\ U_5 = P + 12 \end{cases}$$

$$f(U_{ij}) = \begin{cases} U_6 = P \\ U_{1j} = 3 + 3j & ; j = 1, 2, \dots, \frac{P}{3} \\ U_{2j} = U_{1j} + 1 & ; j = 1, 2, \dots, \frac{P}{3} \\ U_{3j} = U_{1j} + 5 & ; j = 1, 2, \dots, \frac{P}{3} \end{cases}$$

Step 06: Calculate the $|f(U_i) - f(U_{ij})|$ values and assign those integer values for each edge.

Step 07: Verify whether is each vertex assigned an integer from $\{0, 1, \dots, m\}$ and is each edge assigned a distinct integer from the set $\{0, \dots, m\}$ such that the label is equal to the absolute value of the difference of the two vertices the edge connects.

Let's consider a complete graph (K_4) with pendent vertices. The K_4 graph is a simple graph that has four vertices in which each pair of distinct vertices are joined by an edge.

Step 01: Denote the vertices of complete graph U_i as $\{U_1, U_2, U_3, U_4\}$, where $i = 1, 2, 3, 4$

Step 02: Determine the total number of pendant vertices that we want to distribute throughout the vertices of the complete graph (U_1, U_2, U_3, U_4). (Say "P")

Step 03: Distribute the total pendant vertices among $\{U_1, U_2, U_3, U_4\}$ vertices symmetrically.

Step 04: Denote the pendant vertices which are adjacent to $U_1, U_2, U_3,$ and U_4 vertices as $\{U_{1j}, U_{2j}, U_{3j}, U_{4j}\}$ respectively. Where $j = 1, 2, \dots, \frac{P}{4}$

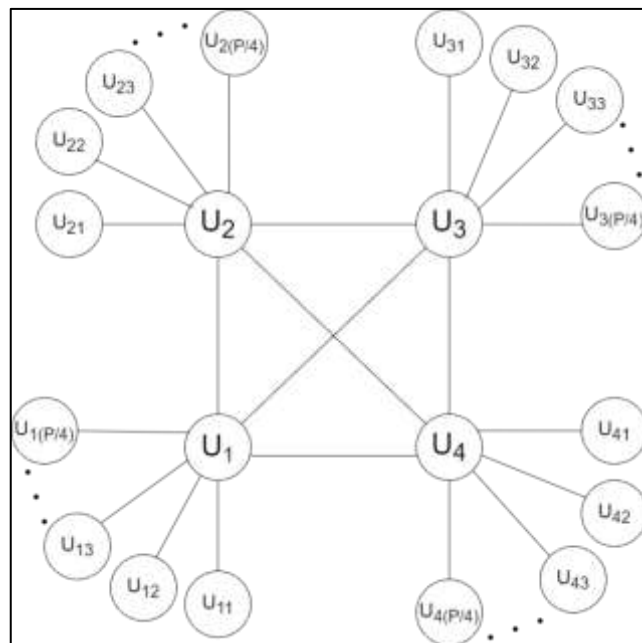


Figure 5. A Complete Graph (K_4) with "P" Number of Pendant Vertices

Step 05: Use the following functions $f(U_i)$ and $f(U_{ij})$ to assign the integer values for vertices in the K_4 graph and corresponding pendent vertices respectively.

Formula 05

$$f(U_i) = \begin{cases} U_1 = 0 \\ U_2 = P + 5 \\ U_3 = P + 6 \\ U_4 = 2 \end{cases}$$

$$f(U_{ij}) = \begin{cases} U_{1j} = U_1 + j + 2 & ; j = 1, 2, \dots, \frac{P}{4} \\ U_{2j} = U_1 \frac{P}{3} + j & ; j = 1, 2, \dots, \frac{P}{4} \\ U_{3j} = U_2 \frac{P}{3} + j + 2 & ; j = 1, 2, \dots, \frac{P}{4} \\ U_{4j} = U_3 \frac{P}{3} + j + 2 & ; j = 1, 2, \dots, \frac{P}{4} \end{cases}$$

Step 06: Calculate the $|f(U_i) - f(U_{ij})|$ values and assign those integer values for each edge.

Step 07: Verify whether is each vertex assigned an integer from $\{0, 1, \dots, m\}$ and is each edge assigned a distinct integer from the set $\{0, \dots, m\}$ such that the label is equal to the absolute value of the difference of the two vertices the edge connects.

Let consider a square pyramid graph with pendant vertices to demonstrate that it is graceful. Square pyramids have eight edges and five vertices.

Step 01: Denote the vertices of square pyramid graph U_i as $\{U_1, U_2, U_3, U_4, U_5\}$, where $i = 1, 2, 3, 4, 5$

Step 02: Determine the total number of pendant vertices that we want to distribute throughout the vertices of the square pyramid graph (U_1, U_2, U_3, U_4). (Say "P")

Step 03: Distribute the total pendant vertices among $\{U_1, U_2, U_3, U_4\}$ vertices symmetrically.

Step 04: Denote the pendant vertices which are adjacent to U_1, U_2, U_3 , and U_4 vertices as $\{U_{1j}, U_{2j}, U_{3j}, U_{4j}\}$ respectively. Where $j = 1, 2, \dots, \frac{P}{4}$

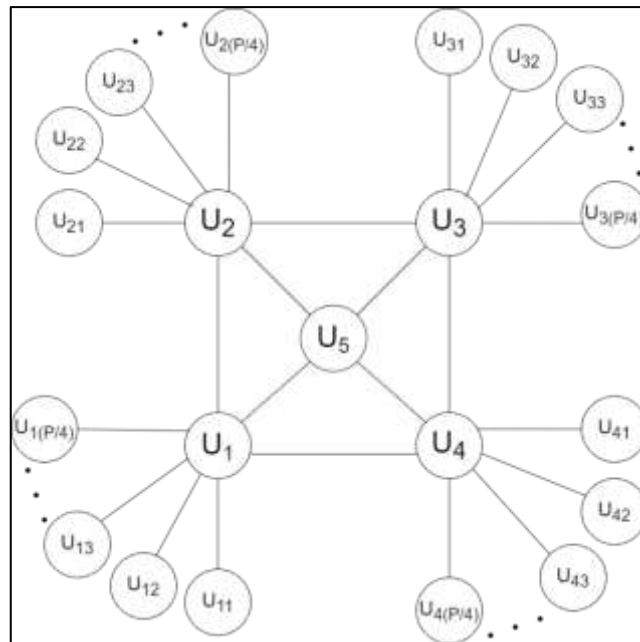


Figure 6. Square Pyramid Graph with “P” Number of Pendant Vertices

Step 05: Use the following functions $f(U_i)$ and $f(U_{ij})$ to assign the integer values for vertices in the square pyramid graph and corresponding pendent vertices respectively.

Formula 06

$$\begin{aligned}
 f(U_i) &= \left\{ \begin{array}{l} U_1 = P + 7 \\ U_2 = P + 6 \\ U_3 = P + 8 \\ U_4 = 3 \\ U_5 = 0 \end{array} \right. \\
 f(U_{ij}) &= \left\{ \begin{array}{l} U_{1j} = 4j \quad ; j = 1, 2, \dots, \frac{P}{4} \\ U_{2j} = U_{1j} + 1 \quad ; j = 1, 2, \dots, \frac{P}{4} \\ U_{3j} = U_{2j} + 1 \quad ; j = 1, 2, \dots, \frac{P}{4} \\ U_{4j} = U_{3j} + 1 \quad ; j = 1, 2, \dots, \frac{P}{4} \end{array} \right.
 \end{aligned}$$

Step 06: Calculate the $|f(U_i) - f(U_{ij})|$ values and assign those integer values for each edge.

Step 07: Verify whether is each vertex assigned an integer from $\{0, 1, \dots, m\}$ and is each edge assigned a distinct integer from the set $\{0, \dots, m\}$ such that the label is equal to the absolute value of the difference of the two vertices the edge connects.

RESULTS AND DISCUSSION

These are some illustrative examples under figures 01, 02, 03, 04, 05 and 06 respectively.

Example 01:

Let's consider C_3 graph with 27 pendant vertices in which each cycle vertices have 9 pendants. That is $P = 27$. By using the above steps and formula 01 we can easily do vertex graceful labeling as follows.

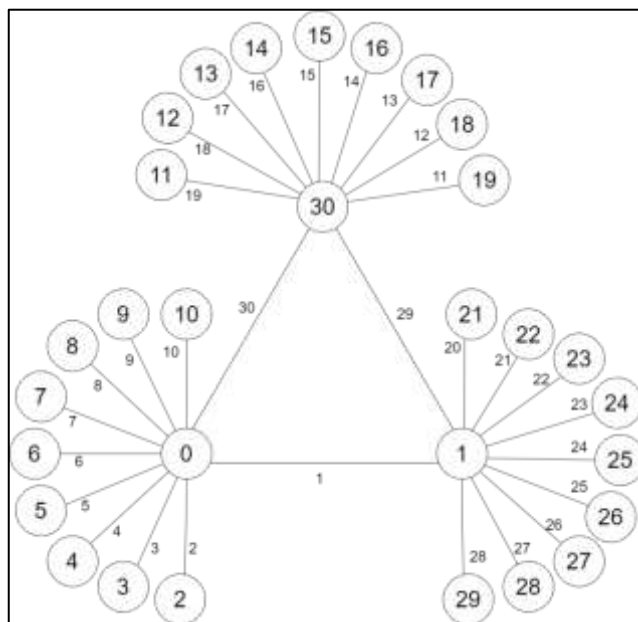


Figure 7. Graceful Labeling for Cyclic Graph (C_3) with 27 Pendant Vertices

According to the above graph, there are 30 edges in the graph. By using formula 01 we have given integer labels from 0 to 30 to all the vertices. The modulus of difference between adjacent vertex labels gives the corresponding edge labels from 1 to 30. Therefore, this is a graceful graph.

Example 02:

Let's consider a tetrahedron graph with 27 pendant vertices in which cycle vertices at the base have 9 pendant vertices. By using the above steps and formula 02 we can do vertex graceful labeling as follows.

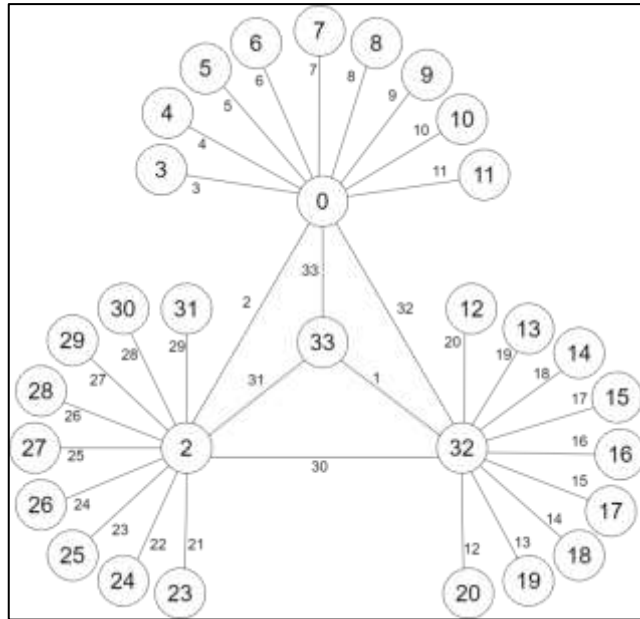


Figure 8. Graceful Labeling for Tetrahedron Graph with 27 Pendant Vertices in Which Cycle Vertices at the Base have 9 Pendant Vertices

In the above tetrahedron graph, there are 33 edges in the graph. By using formula 02 we have given integer labels from 0 to 33 to all the vertices. The modulus of difference between adjacent vertex labels gives the corresponding edge label from 1 to 33. Therefore, this is a graceful graph.

Example 03:

Let's consider a regular graph with 27 pendant vertices. By using the above steps and formula 03 we can do vertex graceful labeling as follows.

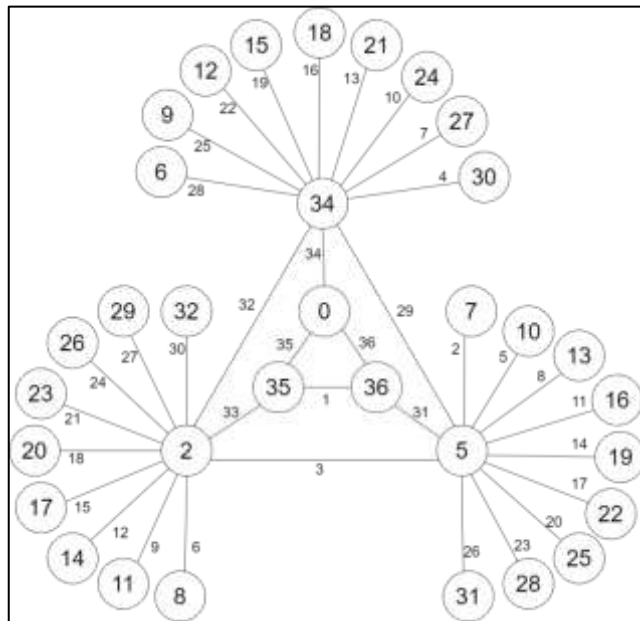


Figure 9. Graceful Labeling for Regular Graph with 27 Number of Pendant Vertices

In the above tetrahedron graph, there are 36 edges in the graph. By using formula 03 we have given integer labels from 0 to 36 to all the vertices. The modulus of difference between adjacent vertex labels gives the corresponding edge label from 1 to 36. Therefore, this is a graceful graph.

Example 04:

Let's consider an octahedron graph with 27 pendant vertices. By using the above steps and formula 04 we can do vertex graceful labeling as follows.

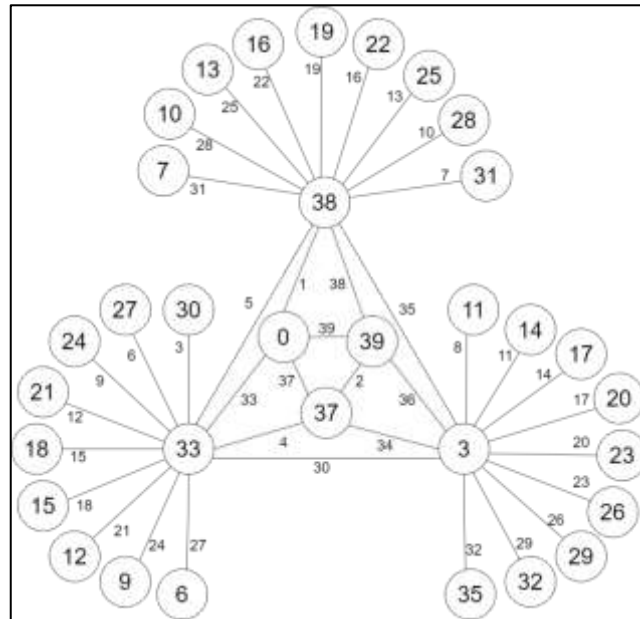


Figure 10. Graceful Labeling for an Octahedron Graph with 27 Number of Pendant Vertices

In the above octahedron graph, there are 39 edges in the graph. By using formula 04 we have given integer labels from 0 to 39 to all the vertices. The modulus of difference between adjacent vertex labels gives the corresponding edge label from 1 to 39. Therefore, this is a graceful graph.

Example 05:

Let's consider a complete graph K_4 with 32 pendant vertices for each $U_1, U_2, U_3,$ and U_4 vertices have 8 pendant vertices. By using the above steps and formula 05 we can do vertex graceful labeling as follows.

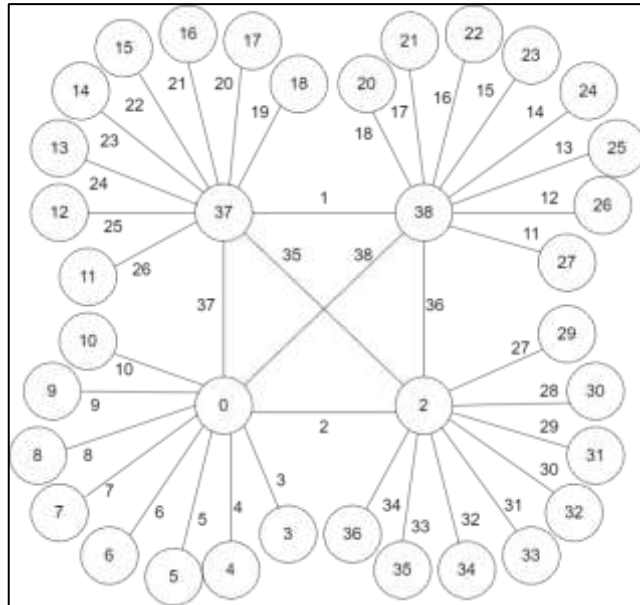


Figure 11. Graceful Labeling for Complete Graph (K_4) with 32 Pendant Vertices

According to the above graph, 38 edges exist. By using formula 03 we have given integer labels from 0 to 38 to all the vertices. The modulus of difference between adjacent vertex labels gives the corresponding edge labels from 1 to 38. Therefore, this is a graceful graph.

Example 06:

Let's consider square pyramid graph with 34 pendant vertices for each U_1, U_2, U_3, U_4 vertices have 8 pendant vertices. By using the above steps and formula 06 we can do vertex graceful labeling as follows.

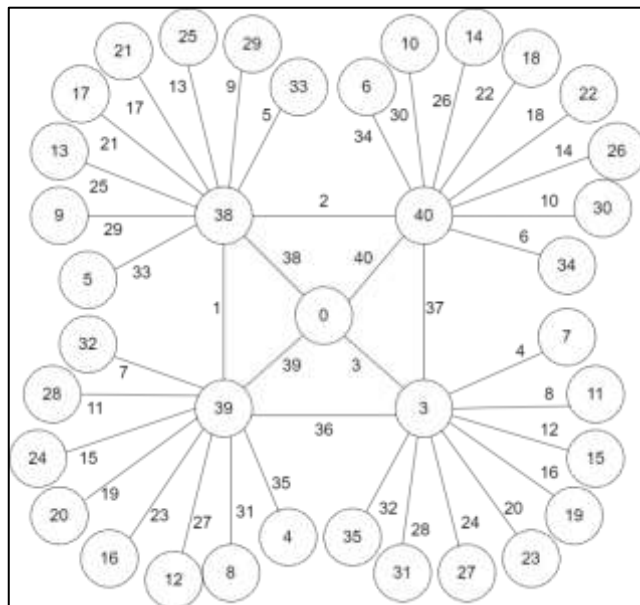


Figure 12. Graceful Labeling for Complete Graph with 36 Pendant Vertices

According to the above graph, 40 edges exist. By using formula 06 we have given integer labels from 0 to 40 to all the vertices. The modulus of difference

between adjacent vertex labels gives the corresponding edge labels from 1 to 40. Therefore, this is a graceful graph.

CONCLUSIONS

The study of graceful labeling for graphs with pendant vertices is a vibrant and active research area. Researchers continue to explore new results, develop innovative techniques, and investigate unsolved problems. This area holds significant potential to contribute to the advancement of graph theory and its applications in diverse fields. This research led to the development of more efficient algorithms. We have successfully implemented and analyzed three algorithms for vertex graceful labeling for the C_3 graph, tetrahedron graph and K_4 graph with pendant vertices. By using those corresponding algorithms, we can easily find graceful labels for any C_3 , tetrahedron graph, and K_4 graph with a finite number of pendants that are symmetrically distributed through the cycle vertices. The result demonstrates and gives evidence that the proposed methods are effective in labeling pendant vertices while maintaining the graceful property of the graph. The method is straightforward to apply and extended to more complex graphs with pendant vertices.

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